Sub-Lorentzian Geometry on Anti-de Sitter Space

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Abstract

Sub-Riemannian Geometry is proved to play an important role in many applications, e.g., Mathematical Physics and Control Theory. Sub-Riemannian Geometry enjoys major differences from the Riemannian being a generalization of the latter at the same time, e.g., geodesics are not unique and may be singular, the Hausdorff dimension is larger than the manifold topological dimension. There exists a large amount of literature developing sub-Riemannian Geometry. However, very few is known about its extension to pseudo-Riemannian analogues. It is natural to begin such a study with some low-dimensional manifolds. Based on ideas from sub-Riemannian geometry we develop sub-Lorentzian geometry over the classical 3-D anti-de Sitter space. Two different distributions of the tangent bundle of anti-de Sitter space yield two different geometries: sub-Lorentzian and sub-Riemannian. We use Lagrangian and Hamiltonian formalisms for both sub-Lorentzian and sub-Riemannian geometries to find geodesics.

Résumé

Il a été prouvé que la Géométrie sub-Riemannienne joue un rôle important dans des nombreuses applications, par ex., Physique Mathématique et Théorie de Contrôle. La Géométrie sub-Riemannienne a des différences considérables par rapport a celle Riemannienne, étant au même temps une généralisation de celle-ci, par ex., les géodésiques ne sont pas uniques et peuvent être singulières, la dimension de Hausdorff est plus grande que la dimension topologique de variété. Il y a une quantité importante de littérature qui développe la Géométrie sub-Riemannienne. Cependant, on connaît très peu sur son extension naturelle aux analogues pseudo Riemanniens. C'est naturel de commencer une telle étude avec des variétés de basse dimension. En se basant sur les idées de la géométrie sub-Riemannienne, on développe la géométrie sub-Lorentzienne sur l'espace anti-de Sitter classique. Deux distributions différentes du faisceau tangent de l'espace d'anti-de Sitter donnent deux géométries différentes: sub-Lorentzienne et sub-Riemannienne. On utilise également les formalismes de Lagrange et d'Hamilton pour les deux géométries, sub-Lorentzienne et sub-Riemanniene, pour trouver les géodésiques.

1. Introduction

Many interesting studies of anticommutative algebras and sub-Riemannian structures may be seen in a general setup of Clifford algebras and spin groups. Among others we distinguish the following example. The unit 3-dimensional sphere S^3 being embedded into the Euclidean space $\mathbb{R}^4$ possesses a clear manifold structure with the Riemannian metric. It is interesting to consider the sphere S^3 as an algebraic object $S^3 = \mathrm{SO}(4)/\mathrm{SO}(3)$ where the group $\mathrm{SO}(4)$ preserves the global Euclidean metric of the ambient space $\mathbb{R}^4$ and $\mathrm{SO}(3)$ preserves the Riemannian metric on S^3. The quotient $\mathrm{SO}(4)/\mathrm{SO}(3)$ can be realized as the group $\mathrm{SU}(2)$ acting on S^3 as on the space of complex vectors z_1, z_2 of unit norm $|z_1|^2 + |z_2|^2 = 1$. It is isomorphic to the group of unit quaternions with the group operation given by the quaternion multiplication. It is natural to make the correspondence between S^3 as a smooth manifold and S^3 as a Lie group acting on this manifold. The corresponding Lie algebra is given by left-invariant vector fields with non-vanishing commutators. This leads to construction of a sub-Riemannian structure on S^3, see [4] (more about sub-Riemannian geometry see, for instance, [11,19,20,21]). The commutation relations for vector fields on the tangent bundle of S^3 come from the non-commutative multiplication for quaternions. Unit quaternions, acting by conjugation on vectors from $\mathbb{R}^3$ (and $\mathbb{R}^4$), define rotation in $\mathbb{R}^3$ (and $\mathbb{R}^4$), thus preserving the positive-definite metric in $\mathbb{R}^4$. At the same time, the Clifford algebra over the vector space $\mathbb{R}^3$ with the standard Euclidean metric gives rise to the spin group $\mathrm{Spin}(3) = \mathrm{SU}(2)$ that acts on the group of unit spinors in the same fashion leaving some positive-definite quadratic form invariant. Two models are equivalent but the latter admits various generalizations. We are primary aimed at switching the Euclidean world to the Lorentzian one and sub-Riemannian geometry to sub-Lorentzian following a simple example similar to the above of a low-dimensional space that leads us to sub-Lorentzian geometry over the pseudohyperbolic space $H^{1,2}$ in $\mathbb{R}^{2,2}$. In General Relativity the simply connected covering manifold of $H^{1,2}$ is called the universal anti-de Sitter space [15,16,22].

We start with some more rigorous explanations. A real Clifford algebra is associated with a vector space V equipped with a quadratic form $Q(\cdot, \cdot)$. The multiplication (let us denote it by $\otimes$) in the Clifford algebra satisfies the relation
$$v \otimes v = -Q(v,v)1,$$
for $v \in V$, where 1 is the unit element of the algebra. We restrict ourselves to $V = \mathbb{R}^3$ with two different quadratic forms:

$$Q_{\mathcal{E}}(v,v) = \mathcal{E}v \cdot v, \quad \mathcal{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$Q(v,v) = Iv \cdot v, \quad I = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The first case represents the standard inner product in the Euclidean space $\mathbb{R}^3$. The second case corresponds to the Lorentzian metric in $\mathbb{R}^3$ given by the diagonal metric tensor with the signature $(-, +, +)$.

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The corresponding Clifford algebras we denote by $\mathrm{Cl}(0,3) = \mathrm{Cl}(3)$ and $\mathrm{Cl}(1,2)$. The basis of the Clifford algebra $\mathrm{Cl}(3)$ consists of the elements

$$\{1, i_1, i_2, i_3, i_1 \otimes i_2, i_1 \otimes i_3, i_2 \otimes i_3, i_1 \otimes i_2 \otimes i_3\}, \quad \text{with} \quad i_1 \otimes i_1 = i_2 \otimes i_2 = i_3 \otimes i_3 = -1.$$

The algebra $\mathrm{Cl}(3)$ can be associated with the space $\mathbb{H} \times \mathbb{H}$, where $\mathbb{H}$ is the quaternion algebra. The basis of the Clifford algebra $\mathrm{Cl}(1,2)$ is formed by

$$\{1, e, i_1, i_2, e \otimes i_1, e \otimes i_2, i_1 \otimes i_2, e \otimes i_1 \otimes i_2\}, \quad \text{with} \quad e \otimes e = 1, \quad i_1 \otimes i_1 = i_2 \otimes i_2 = -1.$$

In this case the algebra is represented by 2×2 complex matrices.

Spin groups are generated by quadratic elements of Clifford algebras. We obtain the spin group $\mathrm{Spin}(3)$ in the case of the Clifford algebra $\mathrm{Cl}(3)$, and the group $\mathrm{Spin}(1,2)$ in the case of the Clifford algebra $\mathrm{Cl}(1,2)$. The group $\mathrm{Spin}(3)$ is represented by the group $\mathrm{SU}(2)$ of unitary 2×2 complex matrices with determinant 1. The elements of $\mathrm{SU}(2)$ can be written as

$$\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, \quad a, b \in \mathbb{C}, \quad |a|^2 + |b|^2 = 1.$$

The group $\mathrm{Spin}(3) = \mathrm{SU}(2)$ forms a double cover of the group of rotations $\mathrm{SO}(3)$. In this case the Euclidean metric in $\mathbb{R}^3$ is preserved under the actions of the group $\mathrm{SO}(3)$. The group $\mathrm{Spin}(3) = \mathrm{SU}(2)$ acts on spinors similarly to how $\mathrm{SO}(3)$ acts on vectors from $\mathbb{R}^3$. Indeed, given an element $R \in \mathrm{SO}(3)$ the rotation is performed by the matrix multiplication RvR^{-1}, where $v \in \mathbb{R}^3$. An element $U \in \mathrm{SU}(2)$ acts over spinors regarded as 2 component vectors $z = (z_1, z_2)$ with complex entries in the same way UzU^{-1}. This operation defines a 'half-rotation' and preserves the positive-definite metric for spinors. Restricting ourselves to spinors of length 1, we get the manifold $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ which is the unit sphere S^3.

Now we turn to the Lorentzian metric and to the Clifford algebra $\mathrm{Cl}(1,2)$. The spin group $\mathrm{Spin}^+(1,2)$ is represented by the group $\mathrm{SU}^+(1,1)$ whose elements are

$$\begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}, \quad a, b \in \mathbb{C}, \quad |a|^2 - |b|^2 = 1.$$

The group $\mathrm{Spin}^+(1,2) = \mathrm{SU}^+(1,1)$ forms a double cover of the group of Lorentzian rotations $\mathrm{SO}(1,2)$ preserving the Lorentzian metric $Q(v,v)$. Acting on spinors, the group $\mathrm{Spin}^+(1,2) = \mathrm{SU}^+(1,1)$ preserves the pseudo-Riemannian metric for spinors. Unit spinors (z_1, z_2), $|z_1|^2 - |z_2|^2 = 1$, are invariant under the actions of the corresponding group $\mathrm{Spin}^+(1,2) = \mathrm{SU}^+(1,1)$. The manifold $H^{1,2} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 - |z_2|^2 = 1\}$ is a 3-dimensional Lorentzian manifold known as a pseudohyperbolic space in Geometry and as the anti-de Sitter space AdS_3 in General Relativity. In fact, anti-de Sitter space is the maximally symmetric, simply connected, Lorentzian manifold of constant negative curvature. It is one of three maximally symmetric cosmological constant solutions to Einstein's field equation: de Sitter space with a positive cosmological constant Λ, anti-de Sitter space with a negative cosmological constant $-\Lambda$, and the flat space. Both de Sitter dS_3 and anti-de Sitter AdS_3 spaces may be treated as non-compact hypersurfaces in the corresponding pseudo-Euclidean spaces $\mathbb{R}^{1,3}$ and $\mathbb{R}^{2,2}$. Sometimes de Sitter space dS_3 or the hypersphere is used as a direct analogue to the sphere S^3 given its positive curvature. However, AdS_3 geometrically is a natural object for us to work with. We reveal the analogy between S^3 and AdS_3 as follows. The group of rotations $\mathrm{SO}(4)$ in the usual Euclidean 4-dimensional space acts as translations on the Euclidean sphere S^3 leaving it invariant. As it has been mentioned at the beginning, the sphere S^3 can be thought of as the Lie group $S^3 = \mathrm{SO}(4)/\mathrm{SO}(3)$ endowed with the group law given by the multiplication of matrices from $\mathrm{SU}(2)$ which is the multiplication law for unit quaternions. The Lie algebra is identified with the left-invariant vector fields from the tangent space at the unity. The tangent bundle admits the natural sub-Riemannian structure and S^3 can be considered as a sub-Riemannian manifold. This geometric object was studied in details in [4]. It appears throughout celestial mechanics in works of Feynman and Vernon who described it in the language of

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two-level systems, in Berry's phase in quantum mechanics or in the Kustaaheimo-Stifel transformation for regularizing binary collision.

Instead of $\mathbb{R}^4$, we consider now the space

$$\mathbb{R}^{2,2} = \{\ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4\ \text{ with a pseudo-metric } dx^2 = -dx_1^2 - dx_2^2 + dx_3^2 + dx_4^2\}.$$

The group $\mathrm{SO}(2,2)$ acting on $\mathbb{R}^{2,2}$ is a direct analog of the rotation group $\mathrm{SO}(4)$ acting on $\mathbb{R}^4$. We consider AdS_3 as a manifold $H^{1,2} = \mathrm{SO}(2,2)/\mathrm{SO}(1,2)$ with the Lorentzian metric induced from $\mathbb{R}^{2,2}$. Sometimes in physics literature, AdS_3 appears as a universal cover of $H^{1,2}$. It is worth to mention that $H^{1,2}$ is a homogeneous non-compact manifold and the group $\mathrm{SO}(2,2)$ acts as an isometry on $H^{1,2}$. The difference between this construction and above mentioned sphere is that S^3 itself is a group, whereas $H^{1,2}$ is not. However, $\mathrm{SO}(2,2)$ can be factorized as $\mathrm{SO}(2,2) = \mathrm{SU}^+(1,1) \times \mathrm{SU}^+(1,1)'$, so $H^{1,2}$ becomes a group manifold for $\mathrm{SU}^+(1,1)$, and topologically they are the same. The group law is defined by the matrix multiplication of elements from $\mathrm{SU}^+(1,1)$. The reader can find more information about the group actions and relation to General Relativity, e. g. [12,17]. Left-invariant vector fields on the tangent bundle are not commutative and this gives us an opportunity to consider an analogue of sub-Riemannian geometry, that is called *sub-Lorentzian* geometry on $\mathrm{SU}^+(1,1)$ (which by abuse of notation, we call the *AdS* group). The geometry of anti-de Sitter space was studied in numerous works, see, for example, [1,5,10,13,18].

Very few is known about extension of sub-Riemannian geometry to its pseudo-Riemannian analogues. The simplest example of a sub-Riemannian structure is provided by the 3-D Heisenberg group. Let us mention that recently Grochowski studied its sub-Lorenzian analogue [7,8]. Our approach deals with non-nilpotent groups over S^3 and AdS_3.

The paper is organized in the following way. In Section 2 we give the precise form of left-invariant vector fields defining sub-Lorentzian and sub-Riemannian structures on anti-de Sitter group. In Sections 3 and 4 the question of existence of smooth horizontal curves in the sub-Lorentzian manifold is studied. The Lagrangian and Hamiltonian formalisms are applied to find sub-Lorentzian geodesics in Sections 5 and 6. Section 7 is devoted to the study of a sub-Riemannian geometry defined by the distribution generated by spacelike vector fields of anti-de Sitter space. In both sub-Lorentzian and sub-Riemannian cases we find geodesics explicitly.

2. Left-invariant vector fields

We consider the AdS group topologically as a 3-dimensional manifold $H^{1,2}$ in $R^{2,2}$

$$H^{1,2} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^{2,2} : \ -x_1^2 - x_2^2 + x_3^2 + x_4^2 = -1\},$$

and as a group $\mathrm{SU}^+(1,1)$ with the group law given by the multiplication of the matrices from $\mathrm{SU}^+(1,1)$. We write $a = x_1 + ix_2$, $b = x_3 + ix_4$, where i is the complex unity. For each matrix $\begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \in \mathrm{SU}^+(1,1)$ we associate its coordinates to the complex vector $p = (a, b)$. Then the multiplication law between $p = (a, b)$ and $q = (c, d)$ written in coordinates is

$$pq = (a, b)(c, d) = (ac + b\bar{d}, ad + b\bar{c}). \tag{2.1}$$

Then, AdS with the multiplication law (2.1) is the Lie group with the unity $(1, 0)$, with the inverse to $p = (a, b)$ element $p^{-1} = (\bar{a}, -b)$, and with the left translation $L_p(q) = pq$. The Lie algebra is associated with the left-invariant vector fields at the identity of the group. To calculate the real left-invariant vector fields, we write the multiplication law (2.1) in real coordinates, setting $c = y_1 + iy_2$, $d = y_3 + iy_4$. Then

$$\begin{aligned}
pq &= (x_1, x_2, x_3, x_4)(y_1, y_2, y_3, y_4) \\
&= (x_1 y_1 - x_2 y_2 + x_3 y_3 + x_4 y_4,\ x_2 y_1 + x_1 y_2 + x_4 y_3 - x_3 y_4, \\
&\quad\ x_3 y_1 + x_4 y_2 + x_1 y_3 - x_2 y_4,\ x_4 y_1 - x_3 y_2 + x_2 y_3 + x_1 y_4).
\end{aligned} \tag{2.2}$$

The tangent map $(L_p)_*$ corresponding to the left translation $L_p(q)$ is

$$(L_p)_* = \begin{bmatrix} x_1 & -x_2 & x_3 & x_4 \\ x_2 & x_1 & x_4 & -x_3 \\ x_3 & x_4 & x_1 & -x_2 \\ x_4 & -x_3 & x_2 & x_1 \end{bmatrix}.$$

The left-invariant vector fields are the left translations of vectors at the unity by the tangent map $(L_p)_*$: $\widetilde{X} = (L_p)_* X(0)$. Letting $X(0)$ be the vectors of the standard basis in $\mathbb{R}^{2,2}$ (that coincides with the Euclidean basis in $\mathbb{R}^4$), we get the left-invariant vector fields

$$\widetilde{X}_1 = x_1 \partial_{x_1} + x_2 \partial_{x_2} + x_3 \partial_{x_3} + x_4 \partial_{x_4},$$
$$\widetilde{X}_2 = -x_2 \partial_{x_1} + x_1 \partial_{x_2} + x_4 \partial_{x_3} - x_3 \partial_{x_4},$$
$$\widetilde{X}_3 = x_3 \partial_{x_1} + x_4 \partial_{x_2} + x_1 \partial_{x_3} + x_2 \partial_{x_4},$$
$$\widetilde{X}_4 = x_4 \partial_{x_1} - x_3 \partial_{x_2} - x_2 \partial_{x_3} + x_1 \partial_{x_4}$$

in the basis $\partial_{x_1}, \partial_{x_2}, \partial_{x_3}, \partial_{x_4}$. Let us introduce the matrices

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then the left-invariant vector fields can be written in the form

$$\widetilde{X}_1 = xU \cdot \nabla_x, \quad \widetilde{X}_2 = xJ \cdot \nabla_x, \quad \widetilde{X}_3 = xE_1 \cdot \nabla_x, \quad \widetilde{X}_4 = xE_2 \cdot \nabla_x,$$

where $x = (x_1, x_2, x_3, x_4)$, $\nabla_x = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3}, \partial_{x_4})$ and "$\cdot$" is the dot-product in $\mathbb{R}^4$. The matrices possess the following properties:

• Anti-commutative rule or the Clifford algebra condition:

$$JE_1 + E_1 J = 0, \quad E_2 E_1 + E_1 E_2 = 0, \quad JE_2 + E_2 J = 0. \tag{2.3}$$

• Non-commutative rule:

$$[\frac{1}{2}J, \frac{1}{2}E_1] = \frac{1}{4}(JE_1 - E_1 J) = \frac{1}{2}E_2, \quad [\frac{1}{2}E_2, \frac{1}{2}E_1] = \frac{1}{2}J, \quad [\frac{1}{2}J, \frac{1}{2}E_2] = -\frac{1}{2}E_1. \tag{2.4}$$

• Transpose matrices:

$$J^T = -J, \quad E_2^T = E_2, \quad E_1^T = E_1. \tag{2.5}$$

• Square of matrices:

$$J^2 = -U, \quad E_2^2 = U, \quad E_1^2 = U. \tag{2.6}$$

As a consequence we obtain
• Product of matrices:

$$JE_1 = E_2, \quad E_2 E_1 = J, \quad JE_2 = -E_1. \tag{2.7}$$

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The inner $\langle \cdot, \cdot \rangle$ product in $\mathbb{R}^{2,2}$ is given by

$$\langle x, y \rangle = \mathcal{I}x \cdot y, \quad \text{with} \quad \mathcal{I} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.8}$$

Given the inner product (2.8) we have

$$\langle x, xE_1 \rangle = \langle x, xJ \rangle = \langle x, xE_2 \rangle = 0, \tag{2.9}$$

$$\langle xJ, xE_1 \rangle = \langle xE_2, xE_1 \rangle = \langle xJ, xE_2 \rangle = 0, \tag{2.10}$$

$$\langle xJ, xJ \rangle = -1, \quad \langle xE_2, xE_2 \rangle = \langle xE_1, xE_1 \rangle = 1. \tag{2.11}$$

The vector field $\widetilde{X}_1$ is orthogonal to AdS. Indeed, if we write AdS as a hypersurface $F(x_1, x_2, x_3, x_4) = -x_1^2 - x_2^2 + x_3^2 + x_4^2 + 1 = 0$, then

$$\frac{dF(c(s))}{ds} = 2\left(-x_1 \frac{dx_1}{ds} - x_2 \frac{dx_2}{ds} + x_3 \frac{dx_3}{ds} + x_4 \frac{dx_4}{ds} \right) = \langle \widetilde{X}_1, \frac{dc(s)}{ds} \rangle = 0$$

for any smooth curve $c(s) = (x_1(s), x_2(s), x_3(s), x_4(s))$ on AdS. From now on we denote the vector field $\widetilde{X}_1$ by N. Observe, that $|N|^2 = \langle N, N \rangle = -1$. Up to certain ambiguity we use the same notation $|\cdot|$ as the norm (whose square is not necessary positive) of a vector and as the absolute value (non-negative) of a real/complex number. Other vector fields are orthogonal to N with respect to the inner product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^{2,2}$:

$$\langle N, \widetilde{X}_2 \rangle = \langle N, \widetilde{X}_3 \rangle = \langle N, \widetilde{X}_4 \rangle = 0.$$

We conclude that the vector fields $\widetilde{X}_2$, $\widetilde{X}_3$, $\widetilde{X}_4$ are tangent to AdS. Moreover, they are mutually orthogonal with

$$|\widetilde{X}_2|^2 = \langle \widetilde{X}_2, \widetilde{X}_2 \rangle = -1, \quad |\widetilde{X}_3|^2 = |\widetilde{X}_4|^2 = 1.$$

We denote the vector field $\widetilde{X}_2$ by T providing time orientation (for the terminology see the end of the present section). The spacelike vector fields $\widetilde{X}_3$ and $\widetilde{X}_4$ will be denoted by X and Y respectively. We conclude that T, X, Y is the basis of the tangent bundle of AdS. In Table 1 the commutative relations between T, X, and Y are presented. We see that if we fix two of the vector fields, then they generate,

Table 1
Commutators of left-invariant vector fields

	T	X	Y
T	0	$2Y$	$-2X$
X	$-2Y$	0	$-2T$
Y	$2X$	$2T$	0

together with their commutators, the tangent bundle of the manifold AdS.

Definition 1 *Let M be a smooth n-dimensional manifold, $\mathcal{D}$ be a smooth k-dimensional, $k < n$, bracket generating distribution on TM, and $\langle \cdot, \cdot \rangle_{\mathcal{D}}$ be a smooth Lorentzian metric on $\mathcal{D}$. Then the triple $(M, \mathcal{D}, \langle \cdot, \cdot \rangle_{\mathcal{D}})$ is called the sub-Lorentzian manifold.*

We deal with two following cases in Sections 3–6 and Section 7 respectively:

1. The horizontal distribution $\mathcal{D}$ is generated by the vector fields T and X: $\mathcal{D} = \text{span}\{T, X\}$. In this case T provides the time orientation and X gives the spatial direction on $\mathcal{D}$. The direction $Y = \frac{1}{2}[T, X]$, orthogonal to the distribution $\mathcal{D}$, is the second spatial direction on the tangent bundle. The metric $\langle \cdot, \cdot \rangle_{\mathcal{D}}$ is given by the restriction of $\langle \cdot, \cdot \rangle$ from $\mathbb{R}^{2,2}$. This case corresponds to the sub-Lorentzian manifold $(AdS, \mathcal{D}, \langle \cdot, \cdot \rangle_{\mathcal{D}})$.

2. The horizontal distribution $\mathcal{D}$ is generated by the vector fields X and Y: $\mathcal{D} = \text{span}\{X, Y\}$. In this case both of the directions are spatial on $\mathcal{D}$. The direction $T = \frac{1}{2}[Y, X]$, orthogonal to the distribution $\mathcal{D}$. In this case, the triple $(AdS, \mathcal{D}, \langle \cdot, \cdot \rangle_{\mathcal{D}})$ is a sub-Riemannian manifold.

The ambient metric with the signature $(-, -, +, +)$ of $\mathbb{R}^{2,2}$ restricted to the tangent bundle $TAdS$ of AdS is the Lorentzian metric with the signature $(-, +, +)$, and therefore, AdS is a Lorentzian manifold. The vector fields T, X, Y form an orthonormal basis of each tangent space $T_p AdS$ at $p \in AdS$. We introduce a time orientation on AdS. A vector $v \in T_p AdS$ is said to be *timelike* if $\langle v, v \rangle < 0$, *spacelike* if $\langle v, v \rangle > 0$ or $v = 0$, and *lightlike* if $\langle v, v \rangle = 0$ and $v \neq 0$. By previous consideration we have T as a timelike vector field and X, Y as spacelike vector fields at each $p \in AdS$. A timelike vector $v \in T_p AdS$ is said to be future-directed if $\langle v, T \rangle < 0$ or past-directed if $\langle v, T \rangle > 0$. A smooth curve $\gamma : [0, 1] \to AdS$ with $\gamma(0) = p$ and $\gamma(1) = q$ is called timelike (spacelike, lightlike) if the tangent vector $\dot{\gamma}(t)$ is timelike (spacelike, lightlike) for any $t \in [0, 1]$. If $\Omega_{p,q}$ is the non-empty set of all timelike, future-directed smooth curves $\gamma(t)$ connecting the points p and q on AdS, then the distance between p and q is defined as

$$\text{dist} := \sup_{\gamma \in \Omega_{p,q}} \int_0^1 \sqrt{-\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle} dt.$$

A geodesic in any manifold M is a curve $\gamma : [0, 1] \to M$ whose vector field is parallel, or equivalently, geodesics are the curves of acceleration zero. A manifold M is called geodesically connected if, given two points $p, q \in M$, there is a geodesic curve $\gamma(t)$ connecting them. Anti-de Sitter space AdS is not geodesically connected, see [9,14].

The concept of causality is important in the study of Lorentz manifolds. We say that $p \in M$ chronologically (causally) precedes $q \in M$ if there is a timelike (non-spacelike) future-directed (if non-zero) curve starting at p and ending at q. For each $p \in M$ we define the chronological future of p as

$$I^+(p) = \{q \in M : \ p \text{ chronologically precedes } q\},$$

and the causal future of p as

$$J^+(p) = \{q \in M : \ p \text{ causally precedes } q\}.$$

The conformal infinity due to Penrose is timelike. One can make analogous definitions replacing 'future' by 'past'.

From the mathematical point of view the spacelike curves have the same right to be studied as timelike or lightlike curves. Nevertheless, the timelike curves and lightlike curves possess an additional physical meaning as the following example shows.

Example 1. Interpreting the x_1-coordinate of AdS as time measured in some inertial frame ($x_1 = t$), the timelike curves represent motions of particles such that

$$\left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dx_3}{dt}\right)^2 < 1.$$

It is assumed that units have been chosen so that 1 is the maximal allowed velocity for a matter particle (the speed of light). Therefore, timelike curves represents motions of matter particles. Timelike geodesics represent motions with constant speed. In addition, the length

$$\tau(\gamma) = \int_0^1 \sqrt{-\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle} \, dt,$$

of a timelike curve $\gamma : [0, 1] \to AdS$ is interpreted as the proper time measured by a particle between events $\gamma(0)$ and $\gamma(1)$.

Lightlike curves represent motions at the speed of light and the lightlike geodesics represent motions along the light rays.

3. Horizontal curves with respect to the distribution $\mathcal{D} = \mathrm{span}\{T, X\}$

Up to Section 7 we shall work with the horizontal distribution $\mathcal{D} = \mathrm{span}\{T, X\}$ and the Lorentzian metric on $\mathcal{D}$, which is the restriction of the metric $\langle \cdot, \cdot \rangle$ from $\mathbb{R}^{2,2}$. We say that an absolutely continuous curve $c(s) : [0, 1] \to AdS$ is *horizontal* if the tangent vector $\dot{c}(s)$ satisfies the relation $\dot{c}(s) = \alpha(s)T(c(s)) + \beta(s)X(c(s))$.

Lemma 1 *A curve $c(s) = (x_1(s), x_2(s), x_3(s), x_4(s))$ is horizontal with respect to the distribution $\mathcal{D} = \mathrm{span}\{T, X\}$, if and only if,*

$$-x_4\dot{x}_1 + x_3\dot{x}_2 - x_2\dot{x}_3 + x_1\dot{x}_4 = 0 \qquad or \qquad \langle xE_2, \dot{c} \rangle = 0. \tag{3.1}$$

PROOF. The tangent vector to the curve $c(s) = (x_1(s), x_2(s), x_3(s), x_4(s))$ written in the left-invariant basis (T, X, Y) admits the form

$$\dot{c}(s) = \alpha T + \beta X + \gamma Y.$$

Then

$$\gamma = \langle \dot{c}, Y \rangle = \mathcal{I}\dot{c} \cdot Y = -x_4\dot{x}_1 + x_3\dot{x}_2 - x_2\dot{x}_3 + x_1\dot{x}_4 = \langle xE_2, \dot{c} \rangle.$$

We conclude that

$$\gamma = 0,$$

if and only if, the condition (3.1) holds. $\square$

In other words, a curve $c(s)$ is horizontal, if and only if, its velocity vector $\dot{c}(s)$ is orthogonal to the missing direction Y. The left-invariant coordinates $\alpha(s)$ and $\beta(s)$ of a horizontal curve $c(s) = (x_1(s), x_2(s), x_3(s), x_4(s))$ are

$$\alpha = \langle \dot{c}, T \rangle = x_2\dot{x}_1 - x_1\dot{x}_2 + x_4\dot{x}_3 - x_3\dot{x}_4 = \langle xJ, \dot{c} \rangle, \tag{3.2}$$

$$\beta = \langle \dot{c}, X \rangle = -x_3\dot{x}_1 - x_4\dot{x}_2 + x_1\dot{x}_3 + x_2\dot{x}_4 = \langle xE_1, \dot{c} \rangle. \tag{3.3}$$

Let us write the definition of the horizontal distribution $\mathcal{D} = \mathrm{span}\{T, X\}$ using the contact form. We define the form $\omega = -x_4dx_1 + x_3dx_2 - x_2dx_3 + x_1dx_4 = \langle xE_2, dx \rangle$. Then,

$$\omega(N) = 0, \quad \omega(T) = 0, \quad \omega(X) = 0, \quad \omega(Y) = 1,$$

and $\ker \omega = \mathrm{span}\{N, T, Y\}$, The horizontal distribution can be defined as follows

$$\mathcal{D} = \{V \in TAdS : \ \omega(V) = 0\}, \quad \text{or} \quad \mathcal{D} = \ker \omega \cap TAdS,$$

where $TAdS$ is the tangent bundle of AdS.

The length $l(c)$ of a horizontal curve $c(s) : \ [0, 1] \to AdS$ is defined by the following formula

$$l(c) = \int_0^1 |\langle \dot{c}(s), \dot{c}(s) \rangle|^{1/2} \, ds.$$

Using the orthonormality of the vector fields T and X, we deduce that

$$l(c) = \int_0^1 \left| -\alpha^2(s) + \beta^2(s) \right|^{1/2} \, ds.$$

We see that the restriction onto the horizontal distribution $\mathcal{D} \subset TAdS$ of the non-degenerate metric $\langle \cdot, \cdot \rangle$ defined on $TAdS$ gives the Lorentzian metric which is non-degenerate. The definitions of timelike (spacelike, lightlike) horizontal vectors $v \in \mathcal{D}_p$ are the same as for the vectors $v \in T_pAdS$. A horizontal curve $c(s)$ is timelike (spacelike, lightlike) if its velocity vector $\dot{c}(s)$ is horizontal timelike (spacelike, lightlike) vector at each point of this curve.

Lemma 2 *Let $\gamma(s) = (y_1(s), y_2(s), y_3(s), y_4(s))$ be a horizontal timelike future-directed (or past-directed) curve and $c(s) = L_p(\gamma(s))$ be its left translation by $p = (p_1, p_2, p_3, p_4)$, $p \in AdS$. Then the curve $c(s)$ is horizontal timelike and future-directed (or past-directed).*

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PROOF. Let us denote by $(c_1(s), c_2(s), c_3(s), c_4(s))$ the coordinates of the curve $c(s)$. Then, by (2.2) we have

$$
\begin{aligned}
c_1(s) &= p_1 y_1(s) - p_2 y_2(s) + p_3 y_3(s) + p_4 y_4(s), \\
c_2(s) &= p_2 y_1(s) + p_1 y_2(s) + p_4 y_3(s) - p_3 y_4(s), \\
c_3(s) &= p_3 y_1(s) + p_4 y_2(s) + p_1 y_3(s) - p_2 y_4(s), \\
c_4(s) &= p_4 y_1(s) - p_3 y_2(s) + p_2 y_3(s) + p_1 y_4(s).
\end{aligned}
\tag{3.4}
$$

Differentiating with respect to s, we calculate the horizontality condition (3.1) for the curve $c(s)$. Since $-p_1^2 - p_2^2 + p_3^2 + p_4^2 = -1$, straightforward simplifications lead to the relation

$$
\langle \dot{c}, Y \rangle = -c_4 \dot{c}_1 + c_3 \dot{c}_2 - c_2 \dot{c}_3 + c_1 \dot{c}_4 = (-p_1^2 - p_2^2 + p_3^2 + p_4^2)(-y_4 \dot{y}_1 + y_3 \dot{y}_2 - y_2 \dot{y}_3 + y_1 \dot{y}_4) = 0,
$$

and the curve γ is horizontal.

Let us show that the curve $c(s)$ is timelike and future-directed provided $\gamma(s)$ is such. We calculate

$$
\langle \dot{c}, T \rangle = c_2 \dot{c}_1 - c_1 \dot{c}_2 + c_4 \dot{c}_3 - c_3 \dot{c}_4 = (p_1^2 + p_2^2 - p_3^2 - p_4^2)(y_2 \dot{y}_1 - y_1 \dot{y}_2 + y_4 \dot{y}_3 - y_3 \dot{y}_4) = \langle \dot{\gamma}, T \rangle
$$

and

$$
\langle \dot{c}, X \rangle = -c_3 \dot{c}_1 - c_4 \dot{c}_2 + c_1 \dot{c}_3 + c_2 \dot{c}_4 = (p_1^2 + p_2^2 - p_3^2 - p_4^2)(-y_3 \dot{y}_1 - y_4 \dot{y}_2 + y_1 \dot{y}_3 + y_2 \dot{y}_4) = \langle \dot{\gamma}, X \rangle
$$

from (3.2), (3.3), and (3.4). Since the horizontal coordinates are not changed, we conclude that the property timelikeness and future-directness is preserved under the left translations. □

In view that the left-invariant coordinates of the velocity vector to a horizontal curve do not change under left translations, we conclude the following analogue of the preceding lemma.

Lemma 3 *Let $\gamma(s) = (y_1(s), y_2(s), y_3(s), y_4(s))$ be a horizontal spacelike (or lightlike) curve and $c(s) = L_p(\gamma(s))$ be its left translation by $p = (p_1, p_2, p_3, p_4)$, $p \in AdS$. Then the curve $c(s)$ is horizontal spacelike (or lightlike).*

4. Existence of smooth horizontal curves on *AdS*

The question of the connectivity by geodesics of two arbitrary points on a Lorentzian manifold is not trivial, because we have to distinguish timelike and spacelike curves. The problem becomes more difficult if we study connectivity for sub-Lorentzian geometry. In the classical Riemannian geometry all geodesics can be found as solutions to the Euler-Lagrange equations and they coincide with the solutions to the corresponding Hamiltonian system obtained by the Legendre transform. In the sub-Riemannian geometry, any solution to the Hamiltonian system is a horizontal curve and satisfies the Euler-Lagrange equations. However, a solution to the Euler-Lagrange equations is a solution to the Hamiltonian system only if it is horizontal.

In the case of sub-Lorentzian geometry we have no information about such a correspondence. As it will be shown in Sections 6 and 7 the solutions to the Hamiltonian system are horizontal. It is a rather expectable fact given the corresponding analysis of sub-Riemannian structures, e. g., on nilpotent groups, see [2,3]. Since $\{T, X, Y = 1/2[T, X]\}$ span the tangent space at each point of AdS the existence of horizontal curves is guaranteed by Chow's theorem [6]. So as the first step, in this section we study connectivity by smooth horizontal curves. The main results states that any two points can be connected by a smooth horizontal curve. A naturally arisen question is whether the found horizontal curve is timelike (spacelike, lightlike)?

First, we introduce a parametrization of AdS and present the horizontality condition and the horizontal coordinates in terms of this parametrisation.

The manifold AdS can be parametrized by

$$x_1 = \cos a \cosh \theta,$$
$$x_2 = \sin a \cosh \theta,$$
$$x_3 = \cos b \sinh \theta,$$
$$x_4 = \sin b \sinh \theta,$$

(4.1)

with $a, b \in (-\pi, +\pi]$, $\theta \in (-\infty, \infty)$. Setting $\psi = a - b$, $\varphi = a + b$, we formulate the following lemma.

Lemma 4 *Let $c(s) = (\varphi(s), \psi(s), \theta(s))$ be a curve on AdS. The curve is horizontal, if and only if,*
$$\dot{\varphi} \cos \psi \sinh 2\theta - 2\dot{\theta} \sin \psi = 0. \tag{4.2}$$
The horizontal coordinates α and β of the velocity vector are
$$\alpha = -\frac{1}{2}(\dot{\varphi} \cosh 2\theta + \dot{\psi}) = -\dot{a} \cosh^2 \theta - \dot{b} \sinh^2 \theta, \tag{4.3}$$
$$\beta = \frac{1}{2}(\dot{\varphi} \sin \psi \sinh 2\theta + 2\dot{\theta} \cos \psi). \tag{4.4}$$

PROOF. Using the parametrisation (4.1) of AdS, we calculate
$$\dot{x}_1 = -\dot{a} \sin a \cosh \theta + \dot{\theta} \cos a \sinh \theta,$$
$$\dot{x}_2 = \dot{a} \cos a \cosh \theta + \dot{\theta} \sin a \sinh \theta,$$
$$\dot{x}_3 = -\dot{b} \sin b \sinh \theta + \dot{\theta} \cos b \cosh \theta,$$
$$\dot{x}_4 = \dot{b} \cos b \sinh \theta + \dot{\theta} \sin b \cosh \theta.$$

(4.5)

Substituting the expressions for x_k and $\dot{x}_k$, $k = 1, 2, 3, 4$, in (3.1), (3.2), and (3.3), in terms of φ, ψ and θ, we get the necessary result. $\square$

We also need the following obvious technical lemma formulated without proof.

Lemma 5 *Given q_0, q_1, $I \in \mathbb{R}$, there is a smooth function $q : [0, 1] \to \mathbb{R}$, such that*
$$q(0) = q_0, \qquad q(1) = q_1, \qquad \int_0^1 q(u) \, du = I.$$

Theorem 1 *Let P and Q be two arbitrary points in AdS. Then there is a smooth horizontal curve joining P and Q.*

PROOF. Let $P = P(\varphi_0, \psi_0, \theta_0)$ and $Q = Q(\varphi_1, \psi_1, \theta_1)$ be coordinates of the points P and Q. In order to find a horizontal curve $c(s)$ we must solve equation (4.2) with the boundary conditions

$$c(0) = P, \quad \text{or} \quad \varphi(0) = \varphi_0, \quad \psi(0) = \psi_0, \quad \theta(0) = \theta_0,$$
$$c(1) = Q, \quad \text{or} \quad \varphi(1) = \varphi_1, \quad \psi(1) = \psi_1, \quad \theta(1) = \theta_1.$$

Assume that $\sin \psi \neq 0$ we rewrite the equation (4.2) as
$$2\dot{\theta} = \dot{\varphi} \cot \psi \sinh 2\theta. \tag{4.6}$$
To simplify matters, let us introduce two new smooth functions $p(s)$ and $q(s)$ by
$$2\theta(s) = \operatorname{arcsinh} p(s), \quad \psi(s) = \operatorname{arccot} q(s),$$
and let the function $\varphi(s)$ is set as $\varphi(s) = \varphi_0 + s(\varphi_1 - \varphi_0)$. Then we will define the smooth functions $p(s)$ and $q(s)$ satisfying the horizontality condition (4.6) for $c = c(s)$. Let $k = \varphi_1 - \varphi_0$. Then equation (4.6) admits the form
$$\frac{\dot{p}(s)}{\sqrt{1 + p^2(s)}} = kp(s)q(s).$$

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Separation of variables leads to the equation

$$\frac{dp}{p\sqrt{1+p^2}} = kq(s)\,ds,$$

that after integrating gives

$$-\operatorname{arctanh}\frac{1}{\sqrt{1+p^2(s)}} = k\Big(\int_0^s q(\tau)\,d\tau + C\Big).$$

To define the constant C, we use the boundary conditions at $s = 0$. Observe that

$$\frac{1}{\sqrt{1+p^2(0)}} = \frac{1}{\cosh 2\theta_0} \quad \text{and} \quad \frac{1}{\sqrt{1+p^2(1)}} = \frac{1}{\cosh 2\theta_1}.$$

Then

$$C = -\frac{1}{k}\operatorname{arctanh}\frac{1}{\cosh 2\theta_0}.$$

Applying the boundary condition at $s = 1$ we find the value of $\int_0^1 q(\tau)\,d\tau$ as

$$\int_0^1 q(\tau)\,d\tau = -\frac{1}{k}\Big(\operatorname{arctanh}\frac{1}{\cosh 2\theta_1} + \operatorname{arctanh}\frac{1}{\cosh 2\theta_0}\Big).$$

Since, moreover, $q(0) = \cot\psi_0$, $q(1) = \cot\psi_1$, Lemma 5 implies the existence of a smooth function $q(s)$ satisfying the above relation.

The function $p(s)$ can be defined by

$$\frac{1}{\sqrt{1+p^2(s)}} = -\tanh\Big[k\int_0^s q(\tau)\,d\tau - \operatorname{arctanh}\frac{1}{\cosh 2\theta_0}\Big].$$

The curve $c(s) = \big(\varphi(s), \psi(s), \theta(s)\big) = \big(\varphi_0 + s(\varphi_1 - \varphi_0), \operatorname{arccot} q(s)), \frac{1}{2}\operatorname{arcsinh} p(s)\big)$ is the desired horizontal curve. $\square$

Remark 1 *Of course, the proof is given for a particular parametrisation by a linear function φ. One may easily modify this proof for an arbitrary smooth function φ obtaining a wider class of smooth horizontal curves.*

Some of the points on AdS can be connected by a curve that maintain one of the coordinate constant.

Theorem 2 *If $P = P(\varphi_0, \psi, \theta_0)$ and $Q = Q(\varphi_1, \psi, \theta_1)$ with*

$$\psi = \operatorname{arccot}\Big(\ln\frac{\tanh\theta_1}{\tanh\theta_0}\Big/(\varphi_0 - \varphi_1)\Big) \tag{4.7}$$

are two points that can be connected, then there is a smooth horizontal curve joining P and Q with the constant ψ-coordinate given by (4.7).

PROOF. Let $c = c(\varphi, \psi, \theta)$ be a horizontal curve with the constant ψ-coordinate. Then it satisfies the equation (4.2) that in this case we write as

$$\cot\psi\ d\varphi = \frac{d(2\theta)}{\sinh 2\theta}.$$

Integrating yields

$$\cot\psi\int_{\theta_0}^\theta d\varphi = \int_{\theta_0}^\theta \frac{d(2\theta)}{\sinh 2\theta} \quad \Rightarrow$$

$$\cot\psi\big(\varphi(\theta) - \varphi(\theta_0)\big) = \ln\tanh\theta - \ln\tanh\theta_0. \tag{4.8}$$

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For $\theta = \theta_1$ we get formula (4.7) for the value of ψ. Solving (4.8) with respect to $\varphi(\theta)$ we get

$$\varphi(\theta) = \varphi_0 + \frac{\ln\left(\tanh\theta/\tanh\theta_0\right)}{\cot\psi}$$

with ψ given by (4.7). Finally, the horizontal curve joining the points P and Q satisfies the equation

$$(\varphi, \psi, \theta) = \left(\varphi_0 + \frac{\ln\left(\tanh\theta/\tanh\theta_0\right)}{\cot\psi}, \psi, \theta\right).$$

$\square$

Upon solving the problem of the connectivity of two arbitrary points by a horizontal curve we are interested in determining its character: timelikeness (spacelikeness or lightlikeness). It is not an easy problem. We are able to present some particular examples showing its complexity. Let us start with the following remark.

Remark 2 *If $P, Q \in AdS$ are two points connectable only by a family of smooth timelike (spacelike, lightlike) curves, then smooth horizontal curves (its existence is known by the preceding theorem) joining P and Q are timelike (spacelike, lightlike).*

Indeed, let $\Omega_{P,Q}$ be a family of smooth timelike (lightlike) curves connecting P and Q. If $\delta(s) \in \Omega_{P,Q}$, then its velocity vector $\dot\delta(s)$ can be written in the left-invariant basis T, X, Y as

$$\dot\delta(s) = \alpha(s)T(\delta(s)) + \beta(s)X(\delta(s)) + \gamma(s)Y(\delta(s))$$

with $\langle\dot\delta(s), \dot\delta(s)\rangle = -\alpha^2 + \beta^2 + \gamma^2 < 0(= 0)$. If moreover, it is horizontal, then $\gamma = 0$. Therefore, $-\alpha^2 + \beta^2 < 0(= 0)$, and the horizontal curve connecting P and Q is timelike (lightlike).

If the points P and Q are connectable only by a family of spacelike curves, then the inequality $-\alpha^2 + \beta^2 > \gamma^2$ holds for them. It implies $-\alpha^2 + \beta^2 > 0$ for a horizontal curve. We conclude that in this case the horizontal curve is still spacelike.

Making use of (4.3) and (4.4) as well as parametrisation (4.1) we calculate the square of the velocity vector for a horizontal curve in terms of the variables φ, ψ, θ as

$$-\alpha^2 + \beta^2 = -\dot\varphi^2 - \dot\psi^2 + 4\dot\theta^2 - 2\dot\varphi\dot\psi\cosh 2\theta. \tag{4.9}$$

We present some particular timelike, spacelike, and lightlike solutions of (4.2).

Example 2. Let $\dot\varphi = 0$. Then, $\varphi \equiv \varphi_0$ is constant. In order to satisfy (4.2) we have two options:

2.1 $\dot\theta = 0 \implies \theta \equiv \theta_0$ is constant. Then $|\dot c|^2 = -\dot\psi^2 \leq 0$. We conclude that all non-constant horizontal curves $c(s) = (\varphi_0, \psi(s), \theta_0)$ are timelike. The projections of these curves onto the (x_1, x_2)- and (x_3, x_4)-planes are circles. All lightlike horizontal curves are only constant ones.

2.2 $\psi = \pi n$, $n \in \mathbb{Z}$. Then $|\dot c|^2 = 4\dot\theta^2 \geq 0$. We conclude that all non-constant horizontal curves $c(s) = (\varphi_0, \pi n, \theta(s))$, $n \in \mathbb{Z}$ are spacelike. The projections of these curves onto the (x_1, x_3)- and (x_2, x_4)-planes are hyperbolas. All lightlike horizontal curves are only constant ones.

Example 3. Let $\dot\varphi \neq 0$. We choose φ as a parameter. Then the square of the norm of the velocity vector is

$$-\alpha^2 + \beta^2 = -1 - \dot\psi^2 + 4\dot\theta^2 - 2\dot\psi\cosh 2\theta, \tag{4.10}$$

where the derivatives are taken with respect to the parameter φ. The horizontality condition becomes

$$2\dot\theta\sin\psi = \cos\psi\sinh 2\theta. \tag{4.11}$$

As in the previous example we consider different cases.

3.1 Suppose $\dot\theta = 0$ and assume that $\theta = \theta_0 \neq 0$. Then the horizontal curves are parametrized by $c(s) = (\varphi, \frac{\pi}{2} + \pi n, \theta_0)$, $n \in \mathbb{Z}$. All these curves are timelike, since $|\dot c|^2 = -1$. There are no lightlike or spacelike horizontal curves.

3.2 If $\theta_0 = 0$, then any curve in the (φ, ψ)-plane is horizontal and timelike since $|\dot{c}|^2 = -(1 + \dot{\psi})^2$.

3.3 Suppose that $\dot{\psi} = 0$ and $\psi \equiv \psi_0 \neq \frac{\pi k}{2}$, $k \in \mathbb{Z}$. Then (4.10) and (4.11) are simplified to

$$- \alpha^2 + \beta^2 = -1 + 4\dot{\theta}^2, \tag{4.12}$$

$$\dot{\theta} = K \sinh 2\theta \quad \text{with} \quad K = \frac{\cot \psi_0}{2}. \tag{4.13}$$

Let $\theta = \theta(\varphi)$ solves equation (4.13). Then the horizontal curve

$$c(s) = (\varphi, \psi_0, \theta(\varphi)) \tag{4.14}$$

is timelike when $|\theta| < \frac{1}{2} \operatorname{arcsinh} \frac{1}{2K}$. If $|\theta| > (=)\frac{1}{2} \operatorname{arcsinh} \frac{1}{2K}$, then the horizontal curve (4.14) is spacelike (lightlike).

Thus any two points $P(\varphi_0, \psi_0, \theta_0)$, $Q(\varphi_1, \psi_1, \theta_0)$, can be connected by a piecewise smooth timelike horizontal curve. This curve consists of straight segments with constant φ-coordinates or with coordinate $\psi = \frac{\pi}{2} + \pi n$, $n \in \mathbb{Z}$. In the case $\theta_0 = 0$, this horizontal curve can be constructed to be smooth.

5. Sub-Lorentzian geodesics

In Lorentzian geometry there are no curves of minimal length because two arbitrary points can be connected by a piecewise lightlike curve. However, there do exist timelike curves with maximal length which are timelike geodesics [14]. By this reason, we are looking for the longest curve among all horizontal timelike ones. It will be given by extremizing the action integral $S = \frac{1}{2} \int_0^1 \left(-\alpha^2(s) + \beta^2(s) \right) ds$ under the non-holonomic constrain $\langle xE_2, \dot{c} \rangle = 0$. The extremal curve will satisfy the Euler-Lagrange system

$$\frac{d}{ds} \frac{\partial L}{\partial \dot{c}} = \frac{\partial L}{\partial c} \tag{5.1}$$

with the Lagrangian

$$L(c, \dot{c}) = \frac{1}{2}(-\alpha^2 + \beta^2) + \lambda(s)\langle xE_2, \dot{c} \rangle.$$

The function $\lambda(s)$ is the Lagrange multiplier function and the values of α and β are given by (3.2) and (3.3). The Euler-Lagrange system (5.1) can be written in the form

$$-\dot{\alpha}x_2 - \dot{\beta}x_3 = 2(\alpha\dot{x}_2 + \beta\dot{x}_3 - \lambda\dot{x}_4) - \dot{\lambda}x_4,$$

$$\dot{\alpha}x_1 - \dot{\beta}x_4 = 2(-\alpha\dot{x}_1 + \beta\dot{x}_4 + \lambda\dot{x}_3) + \dot{\lambda}x_3,$$

$$-\dot{\alpha}x_4 + \dot{\beta}x_1 = 2(\alpha\dot{x}_4 - \beta\dot{x}_1 - \lambda\dot{x}_2) - \dot{\lambda}x_2,$$

$$\dot{\alpha}x_3 + \dot{\beta}x_2 = 2(-\alpha\dot{x}_3 - \beta\dot{x}_2 + \lambda\dot{x}_1) + \dot{\lambda}x_4.$$

for the extremal curve $c(s) = (x_1(s), x_2(s), x_3(s), x_4(s))$. Multiplying these equations by x_2, $-x_1$, $-x_4$, x_3, respectively and then, summing them up we obtain

$$-\dot{\alpha} = 2(-\alpha\langle \dot{c}, N \rangle - \beta\langle \dot{c}, Y \rangle - \lambda\beta) = -2\lambda\beta$$

because $\langle \dot{c}, Y \rangle = \langle \dot{c}, N \rangle = 0$. Now, multiplying the equations by x_3, x_4, x_1, x_2, respectively and then, summing them up we get

$$-\dot{\beta} = 2(\alpha\langle \dot{c}, Y \rangle + \beta\langle \dot{c}, N \rangle + \lambda\alpha) = 2\lambda\alpha$$

in a similar way. The values of α and β are concluded to satisfy the system

$$\dot{\alpha}(s) = 2\lambda\beta(s),$$
$$\dot{\beta}(s) = 2\lambda\alpha(s). \tag{5.2}$$

Case $\lambda(s) = 0$. In the Riemannian geometry the Schwartz inequality allows us to define the angle ϑ between two vectors v and w as a unique number $0 \leq \vartheta \leq \pi$, such that

$$\cos \vartheta = \frac{v \cdot w}{|v||w|}.$$

There is an analogous result in Lorentzian geometry which is formulated as follows.

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Proposition 1 *[14] Let v and w be timelike vectors. Then,*
1. *$|\langle v, w\rangle| \geq |v||w|$ where the equality is attained if and only if v and w are collinear.*
2. *If $\langle v, w\rangle < 0$, there is a unique number $\vartheta \geq 0$, called the hyperbolic angle between v and w, such that*

$$\langle v, w\rangle = -|v||w|\cosh\vartheta.$$

Theorem 3 *The family of timelike future-directed horizontal curves contains horizontal timelike future-directed geodesics $c(s)$ with the following properties*
1. *The length $|\dot{c}|$ is constant along the geodesic.*
2. *The inner products $\langle T, \dot{c}\rangle = \alpha$, $\langle X, \dot{c}\rangle = \beta$, $\langle Y, \dot{c}\rangle = 0$ are constant along the geodesic.*
3. *The hyperbolic angle between the horizontal time vector field T and the velocity vector $\dot{c}$ is constant.*

PROOF. The system (5.2) implies

$$\dot{\alpha}(s) = 0 \qquad \dot{\beta}(s) = 0.$$

The existence of a geodesic follows from the general theory of ordinary differential equations, employing, for example, the parametrisation given for α, β, γ in the preceding section. Since the horizontal coordinates $\alpha(s)$ and $\beta(s)$ are constant along the curve c we conclude that c is geodesic. We denote by α and β its respective horizontal coordinates.

The length of the velocity vector $\dot{c}$ is $|-\alpha^2 + \beta^2|^{1/2}$ and it is constant along the geodesic.

The second statement is obvious. Since $c(s)$ is a future-directed geodesic, we have $\langle T, \dot{c}\rangle < 0$, and

$$\cosh(\angle T, \dot{c}) = -\frac{\langle T, \dot{c}\rangle}{|T||\dot{c}|} = \frac{-\alpha}{\sqrt{|-\alpha^2 + \beta^2|}} \quad \text{is constant.}$$

$\square$

Case $\lambda(s) \neq 0$. We continue to study the extremals given by the solutions of the Euler-Lagrange equation (5.1).

Lemma 6 *Let $c(s)$ be a timelike future-directed solution of the Euler-Lagrange system (5.1) with $\lambda(s) \neq 0$. Then,*
1. *The length $|-\alpha^2(s) + \beta^2(s)|^{1/2}$ of the velocity vector $\dot{c}(s)$ is constant along the solution.*
2. *The hyperbolic angle between the curve $c(s)$ and the integral curve of the time vector field T is given by*

$$\vartheta = \angle(\dot{c}, T) = -2\Lambda(s) + \theta_0,$$

where Λ is the primitive of λ.

PROOF. Multiplying the first equation of (5.2) by α, the second one by β and subtracting, we deduce that $\alpha\dot{\alpha} - \beta\dot{\beta} = 0$. This implies that $-\alpha^2 + \beta^2 = \langle\dot{c}, \dot{c}\rangle$ is constant. The horizontal solution is timelike if the initial velocity vector is timelike. The first assertion is proved.

Set $r = \sqrt{|-\alpha^2 + \beta^2|}$. Using the hyperbolic functions we write

$$\alpha(s) = -r\cosh\theta(s), \qquad \beta(s) = r\sinh\theta(s).$$

Substituting α and β in (5.2), we have

$$\dot{\theta}(s) = -2\lambda(s).$$

Denote $\Lambda(s) = \int_0^s \lambda(s)\, ds$ and write the solution of the latter equation as $\theta = -2\Lambda(s) + \theta_0$. Thus,

$$\alpha(s) = -r\cosh(-2\Lambda(s) + \theta_0), \quad \beta(s) = r\sinh(-2\Lambda(s) + \theta_0). \qquad (5.3)$$

In order to find the value of the constant θ_0 we put $s = 0$ and get $\theta_0 = \operatorname{arctanh}\frac{\beta(0)}{\alpha(0)}$.

Let $c(s)$ be a horizontal timelike future-directed solution of (5.1). Then $\langle\dot{c}, T\rangle < 0$ and

$$\alpha = \langle\dot{c}, T\rangle = -|\dot{c}||T|\cosh\vartheta = -r\cosh(\angle(\dot{c}, T)).$$

Comparing with (5.3) finishes the proof of the theorem. $\square$

There is no counterpart of Proposition 1 for spacelike vectors. Nevertheless, we obtain the following analogue of Lemma 6 .

Lemma 7 *Let $c(s)$ be a spacelike solution of the Euler-Lagrange system* (5.1) *with $\lambda(s) \neq 0$. Then,*
1. The length of the velocity vector $\dot{c}(s)$ is constant along the solution;
2. The horizontal coordinates are expressed by (5.3).
As the next step, we shall study the function $\Lambda(s)$. First, let us prove some useful facts.

Proposition 2 *Let $c(s) = (x_1(s), x_2(s), x_3(s), x_4(s))$ be a horizontal timelike (spacelike) curve. Then,*
1. $-\dot{x}_1^2(s) - \dot{x}_2^2(s) + \dot{x}_3^2(s) + \dot{x}_4^2(s) = -\alpha^2(s) + \beta^2(s)$;
2. $\ddot{c} = a(s)T + b(s)X + \omega(s)Y + w(s)N$, with $a = \dot{\alpha}$, $b = \dot{\beta}$, $\omega = 0$, $w = \alpha^2 - \beta^2$.

PROOF. Let us write the coordinates of $\dot{c}(s)$ in the basis T, X, Y, N as

$$\dot{c}(s) = \alpha(s)T + \beta(s)X + \gamma(s)Y + \delta(s)N,$$

where

$$\alpha = \langle \dot{c}, T \rangle = x_2\dot{x}_1 - x_1\dot{x}_2 + x_4\dot{x}_3 - x_3\dot{x}_4,$$

$$\beta = \langle \dot{c}, X \rangle = -x_3\dot{x}_1 - x_4\dot{x}_2 + x_1\dot{x}_3 + x_2\dot{x}_4,$$

$$0 = \gamma = \langle \dot{c}, Y \rangle = x_4\dot{x}_1 - x_3\dot{x}_2 + x_2\dot{x}_3 - x_1\dot{x}_4,$$

$$0 = \delta = \langle \dot{c}, N \rangle = -x_1\dot{x}_1 - x_2\dot{x}_2 + x_3\dot{x}_3 + x_4\dot{x}_4.$$

By the direct calculation we get

$$-\alpha^2 + \beta^2 = -\alpha^2 - \delta^2 + \beta^2 + \gamma^2 = -\dot{x}_1^2 - \dot{x}_2^2 + \dot{x}_3^2 + \dot{x}_4^2.$$

In order to prove the second statement of the proposition we calculate

$$\dot{\alpha} = x_2\ddot{x}_1 - x_1\ddot{x}_2 + x_4\ddot{x}_3 - x_3\ddot{x}_4 = \langle \ddot{c}, T \rangle = a,$$

$$\dot{\beta} = -x_3\ddot{x}_1 - x_4\ddot{x}_2 + x_1\ddot{x}_3 + x_2\ddot{x}_4 = \langle \ddot{c}, X \rangle = b.$$

Differentiating the horizontality condition (3.1), we find

$$0 = \frac{d}{ds}\langle \dot{c}, Y \rangle = \frac{d}{ds}\left(x_4\dot{x}_1 - x_3\dot{x}_2 + x_2\dot{x}_3 - x_1\dot{x}_4 \right) = x_4\ddot{x}_1 - x_3\ddot{x}_2 + x_2\ddot{x}_3 - x_1\ddot{x}_4 = \langle \ddot{c}, Y \rangle = \omega.$$

Then,

$$0 = \frac{d}{ds}\langle \dot{c}, N \rangle = \frac{d}{ds}\left(-x_1\dot{x}_1 - x_2\dot{x}_2 + x_3\dot{x}_3 + x_4\dot{x}_4 \right) = -x_1\ddot{x}_1 - x_2\ddot{x}_2 + x_3\ddot{x}_3 + x_4\ddot{x}_4$$
$$+ (-\dot{x}_1^2 - \dot{x}_2^2 + \dot{x}_3^2 + \dot{x}_4^2) = \langle \ddot{c}, N \rangle + (-\alpha^2 + \beta^2) = w - \alpha^2 + \beta^2,$$

by the first statement. The proof is finished. $\square$

Theorem 4 *The Lagrange multiplier $\lambda(s)$ is constant along the horizontal timelike (spacelike, lightlike) solution of the Euler-Lagrange system* (5.1).

PROOF. We consider the equivalent Lagrangian function $\widehat{L}(x, \dot{x})$, changing the length function $-\alpha^2 + \beta^2$ to $-\dot{x}_1^2 - \dot{x}_2^2 + \dot{x}_3^2 + \dot{x}_4^2$. The solutions of the Euler-Lagrange system for both Lagrangians give the same curve. Thus, the new Lagrangian is

$$\widehat{L}(x, \dot{x}) = \frac{1}{2}\left(-\dot{x}_1^2 - \dot{x}_2^2 + \dot{x}_3^2 + \dot{x}_4^2 \right) + \lambda(s)\left(\dot{x}_1 x_4 - \dot{x}_4 x_1 - \dot{x}_2 x_3 + \dot{x}_3 x_2 \right).$$

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The corresponding Euler-Lagrange system is

$$
\begin{aligned}
-\ddot{x}_1 &= -\dot{\lambda}x_4 - 2\lambda\dot{x}_4, \\
-\ddot{x}_2 &= \dot{\lambda}x_3 + 2\lambda\dot{x}_3, \\
\ddot{x}_3 &= -\dot{\lambda}x_2 - 2\lambda\dot{x}_2, \\
\ddot{x}_4 &= -\dot{\lambda}x_1 + 2\lambda\dot{x}_1.
\end{aligned}
$$

We multiply the first equation by $-x_4$, the second equation by x_3, the third one by x_2, and the last one by $-x_1$, finally, sum them up. This yields

$$
\ddot{x}_1 x_4 - \ddot{x}_2 x_3 + \ddot{x}_3 x_3 - \ddot{x}_4 x_1 = \dot{\lambda}(x_4^2 + x_3^2 - x_2^2 - x_1^2) + 2\lambda(\dot{x}_4 x_4 + \dot{x}_3 x_3 - \dot{x}_2 x_2 - \dot{x}_1 x_1) \quad \Rightarrow
$$

$$
\langle \ddot{c}, Y \rangle = -\dot{\lambda} + 2\lambda\langle \dot{c}, N \rangle \quad \Rightarrow \quad \dot{\lambda} = 0.
$$

We conclude that λ is constant along the solution. $\square$

We see from the proof of Lemma 6 that the function $\Lambda(s)$ is just a linear function. This leads to the following property of horizontal timelike future-directed solutions of the Euler-Lagrange system (5.1).

Corollary 1 *If $c(s)$ is a horizontal timelike future-directed solution of* (5.1)*, then the hyperbolic angle between its velocity and the time vector field T increases linearly in s.*

6. Hamiltonian formalism

The sub-Laplacian, which is the sum of the squares of the horizontal vector fields plays the fundamental role in sub-Riemannian geometry. The counterpart of the sub-Laplacian in the Lorentz setting is the operator

$$
\mathcal{L} = \frac{1}{2}(-T^2 + X^2) = \frac{1}{2}\Big(- \big(-x_2\partial_{x_1} + x_1\partial_{x_2} + x_4\partial_{x_3} - x_3\partial_{x_4}\big)^2
$$
$$
+ \big(x_3\partial_{x_1} + x_4\partial_{x_2} + x_1\partial_{x_3} + x_2\partial_{x_4}\big)^2\Big). \tag{6.1}
$$

Then the Hamiltonian function corresponding to the operator (6.1) is

$$
\begin{aligned}
H(x,\xi) &= \frac{1}{2}\Big(- \big(-x_2\xi_1 + x_1\xi_2 + x_4\xi_3 - x_3\xi_4\big)^2 + \big(x_3\xi_1 + x_4\xi_2 + x_1\xi_3 + x_2\xi_4\big)^2\Big) \\
&= \frac{1}{2}\big(-\tau^2 + \varsigma^2\big),
\end{aligned} \tag{6.2}
$$

where we use the notations $\xi_k = \partial_{x_k}$, $\tau = -x_2\xi_1 + x_1\xi_2 + x_4\xi_3 - x_3\xi_4$, and $\varsigma = x_3\xi_1 + x_4\xi_2 + x_1\xi_3 + x_2\xi_4$. There are close relations between the solutions of the Euler-Lagrange equation and the solutions of the Hamiltonian system

$$
\dot{x} = \frac{\partial H}{\partial \xi}, \qquad \dot{\xi} = -\frac{\partial H}{\partial x}.
$$

The solutions of the Euler-Lagrange system (5.1) coincide with the projection of the solutions of the Hamiltonian system onto the Riemannian manifold. In the sub-Riemannian case the solutions coincide, if and only if, the solution of the Euler-Lagrange system is a horizontal curve. We are interested in relations of the solutions of these two systems in our situation. The Hamiltonian system admits the form

$$
\begin{cases}
\dot{x} = \dfrac{\partial H}{\partial \xi} = -\tau x J + \varsigma x E_1, \\[2mm]
\dot{\xi} = -\dfrac{\partial H}{\partial x} = -\tau \xi J - \varsigma \xi E_1.
\end{cases} \tag{6.3}
$$

Lemma 8 *The solution of the Hamiltonian system* (6.3) *is a horizontal curve and*

$$\tau = \alpha, \qquad \varsigma = \beta, \tag{6.4}$$

where α and β are given by (3.2) *and* (3.3) *respectively.*

PROOF. Let $c(s) = \big(x_1(s), x_2(s), x_3(s), x_4(s)\big)$ be a solution of (6.3). In order to prove its horizontality we need to show that the inner product $\langle \dot{x}, xE_2 \rangle$ vanishes. We substitute $\dot{x}$ from (6.3) and get

$$\langle \dot{x}, xE_2 \rangle = -\tau \langle xJ, xE_2 \rangle + \varsigma \langle xE_1, xE_2 \rangle = 0$$

by (2.10).

Using the first line in the Hamiltonian system and the definitions of horizontal coordinates (3.2) and (3.3), we get

$$\alpha = \langle \dot{x}, xJ \rangle = -\tau \langle xJ, xJ \rangle + \varsigma \langle xE_1, xJ \rangle = \tau,$$

$$\beta = \langle \dot{x}, xE_1 \rangle = -\tau \langle xJ, xE_1 \rangle + \varsigma \langle xE_1, xE_1 \rangle = \varsigma$$

from (2.10) and (2.11). $\square$

6.1. *Geodesics with constant horizontal coordinates*

Lemma 8 implies the following form of the Hamiltonian system (6.3)

$$
\begin{aligned}
\dot{x}_1 &= -\alpha(-x_2) + \beta x_3, \\
\dot{x}_2 &= -\alpha x_1 + \beta x_4, \\
\dot{x}_3 &= -\alpha x_4 + \beta x_1, \\
\dot{x}_4 &= -\alpha(-x_3) + \beta x_2,
\end{aligned}
\tag{6.5}
$$

with constant α and β.

6.1.1. *Timelike case*

In this section we are aimed at finding geodesics corresponding to the extremals (Section 5) with constant horizontal coordinates α and β giving the vanishing value to the Lagrangian multiplier λ. We give an explicit picture for the base point $(1, 0, 0, 0)$. Left shifts transport it to any other point of AdS. Without lost of generality, let us assume that $-\alpha^2 + \beta^2 = -1$, $\alpha = \cosh \psi$, $\beta = \sinh \psi$, where ψ is a constant.

The Hamiltonian system (6.5) written for constant α and β is reduced to a second-order differential equation

$$\ddot{x}_k = -x_k, \quad k = 1, \dots 4. \tag{6.6}$$

The general solution is given in the trigonometric basis as $x_k = A_k \cos s + B_k \sin s$. The initial condition $x(0) = (1, 0, 0, 0)$ defines the coefficients A_k by $A_1 = 1$, $A_2 = A_3 = A_4 = 0$. Returning back to the first-order system (6.5) we calculate the coefficients B_k as $B_1 = 0$, $B_2 = -\alpha$, $B_3 = \beta$, $B_4 = 0$. Finally, the solution is

$$x_1 = \cos s, \quad x_2 = -\cosh \psi \sin s, \quad x_3 = \sinh \psi \sin s, \quad x_4 \equiv 0. \tag{6.7}$$

These timelike geodesics are closed. Varying ψ they sweep out the one-sheet hyperboloid $x_1^2 + x_2^2 - x_3^2 = 1$ in $\mathbb{R}^3$.

Let us calculate the *vertical line* Γ, the line corresponding to the vanishing horizontal velocity (α, β) and with the constant value $\gamma = 1$, passing the base point $(1, 0, 0, 0)$. Its parametric representation $\Gamma = \Gamma(s)$ satisfies the system

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$$\alpha = x_2\dot{x}_1 - x_1\dot{x}_2 + x_4\dot{x}_3 - x_3\dot{x}_4 \quad = 0,$$
$$\beta = -x_3\dot{x}_1 - x_4\dot{x}_2 + x_1\dot{x}_3 + x_2\dot{x}_4 = 0,$$
$$\gamma = x_4\dot{x}_1 - x_3\dot{x}_2 + x_2\dot{x}_3 - x_1\dot{x}_4 \quad = 1,$$
$$\delta = x_1\dot{x}_1 + x_2\dot{x}_2 - x_3\dot{x}_3 - x_4\dot{x}_4 \quad = 0.$$

The discriminant of this system calculated with respect to the derivatives as variables is (-1), and we reduce the system to a simple one

$$\dot{x}_1 = -x_4, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = x_2, \quad \dot{x}_4 = -x_1,$$

with the initial condition $\Gamma(0) = x(0) = (1,0,0,0)$. The solution is

$$\Gamma(s) = (\cosh s, 0, 0, -\sinh s).$$

The vertical line (hyperbola) Γ meets the surface (6.7) at the point (1,0,0,0) orthogonally with respect to the scalar product in $\mathbb{R}^{2,2}$. Comparing this picture with the classical sub-Riemannian case of the Heisenberg group, we observe that in the Heisenberg case all straight line geodesics lie on the horizontal plane $\mathbb{R}^2$ and the center is the third vertical axis. In our case the surface (6.7) corresponds to the horizontal plane, timelike geodesics correspond to the straight line Heisenberg geodesics, and Γ corresponds to the vertical center.

6.1.2. *Spacelike/lightlike case*

Again we consider constant horizontal coordinates α and β, and let us assume that $-\alpha^2 + \beta^2 = 1$, $\alpha = \sinh\psi$, $\beta = \cosh\psi$, where ψ is a constant.

The Hamiltonian system (6.5) is reduced to the second-order differential equation

$$\ddot{x}_k = x_k, \quad k = 1, \ldots 4. \tag{6.8}$$

Arguing as in the previous case we deduce the solution passing the point (1,0,0,0) as

$$x_1 = \cosh s, \quad x_2 = -\sinh\psi\sinh s, \quad x_3 = \cosh\psi\sinh s, \quad x_4 \equiv 0. \tag{6.9}$$

These non-closed spacelike geodesics sweep the same hyperboloid of one sheet in $\mathbb{R}^3$. The vertical line Γ meets orthogonally each spacelike geodesic on this hyperboloid at the point (1,0,0,0).

In the lightlike case $\alpha^2 = \beta^2 = 1$ the Hamiltonian system (6.5) has a linear solution given by

$$x_1 \equiv 1, \quad x_2 = -\alpha s, \quad x_3 = \beta s, \quad x_4 \equiv 0,$$

which are two straight lines on the hyperboloid, and again Γ meets them orthogonally at the unique point (1,0,0,0).

6.2. *Geodesics with non-constant horizontal coordinates.*

If the horizontal coordinates are not constant, then we must solve the Hamiltonian system generated by the Hamiltonian (6.2).

Fix the initial point $x^{(0)} = (1,0,0,0)$. We shall give two approaches to solve this Hamiltonian system based on a solution in Cartesian coordinates and on a parametrization of AdS.

Solution in the Cartesian coordinates. It is convenient to introduce auxiliary phase functions

$$u_1 = x_1 + x_2, \quad u_2 = x_1 - x_2, \quad u_3 = x_3 + x_4, \quad u_4 = x_3 - x_4,$$

and momenta

$$\psi_1 = \xi_1 + \xi_2, \quad \psi_2 = \xi_1 - \xi_2, \quad \psi_3 = \xi_3 + \xi_4, \quad \psi_4 = \xi_3 - \xi_4.$$

Then the Hamiltonian (6.2) admits the form $H = (-u_4\psi_2 + u_1\psi_3)(u_3\psi_1 - u_2\psi_4)$, and yields the Hamiltonian system

$$\dot{u}_1 = u_3(-u_4\psi_2 + u_1\psi_3), \qquad u_1(0) = 1,$$
$$\dot{u}_2 = -u_4(u_3\psi_1 - u_2\psi_4), \qquad u_2(0) = 1,$$
$$\dot{u}_3 = u_1(u_3\psi_1 - u_2\psi_4), \qquad u_3(0) = 0,$$
$$\dot{u}_4 = -u_2(-u_4\psi_2 + u_1\psi_3), \qquad u_4(0) = 0,$$

(6.10)

for positions and

$$\dot{\psi}_1 = -\psi_3(u_3\psi_1 - u_2\psi_4), \qquad \psi_1(0) = A,$$
$$\dot{\psi}_2 = \psi_4(-u_4\psi_2 + u_1\psi_3), \qquad \psi_2(0) = B,$$
$$\dot{\psi}_3 = -\psi_1(-u_4\psi_2 + u_1\psi_3), \qquad \psi_3(0) = C,$$
$$\dot{\psi}_4 = \psi_2(u_3\psi_1 - u_2\psi_4), \qquad \psi_4(0) = D,$$

(6.11)

for momenta with some real constants A, B, C, and D. For τ and ς constant we get simple solutions mentioned in the previous section. We see that the system (6.10–6.11) has the first integrals

$$u_1\psi_1 + u_3\psi_3 = A,$$
$$u_2\psi_2 + u_4\psi_4 = B,$$
$$u_2\psi_3 - u_4\psi_1 = C,$$
$$u_1\psi_4 - u_3\psi_2 = D,$$

and in addition, we normalize $\psi(0)$ so that the trajectories belong to AdS: $u_1u_2 + u_3u_4 = 1$, and the Hamiltonian $H = -1$ in the timelike case, in particular, the latter implies $CD = 1$. Then we can deduce the momenta as

$$\psi_1 = Au_2 - Cu_3,$$
$$\psi_2 = Bu_1 - Du_4,$$
$$\psi_3 = Cu_1 + Au_4,$$
$$\psi_4 = Du_2 + Bu_3.$$

Let us set the functions $p = u_4/u_1$ and $q = u_3/u_2$. Then substituting function ψ in (6.10), we get

$$\dot{p} = -(Dp^2 + (A - B)p + 1/D), \quad p(0) = 0,$$
$$\dot{q} = -(Cq^2 - (A - B)q + 1/C), \quad q(0) = 0.$$

The cases of the discriminant give the following options. Solving these equations for $|A - B| > 2$, we obtain

$$p(s) = \frac{2}{D} \frac{1 - e^{-s\sqrt{(B-A)^2 - 4}}}{(B - A - \sqrt{(B-A)^2 - 4}) - (B - A + \sqrt{(B-A)^2 - 4})e^{-s\sqrt{(B-A)^2 - 4}}},$$

$$q(s) = \frac{2D(1 - e^{-s\sqrt{(A-B)^2 - 4}})}{(A - B - \sqrt{(A-B)^2 - 4}) - (A - B + \sqrt{(A-B)^2 - 4})e^{-s\sqrt{(A-B)^2 - 4}}}.$$

Next we use the relation $\dot{u}_1 = -\frac{u_3}{u_2}\dot{u}_4$. Then, $\dot{u}_1(pq + 1) = -\dot{p}qu_1$, and finally,

$$u_1(s) = \exp\int_0^s \frac{-\dot{p}(t)q(t)}{p(t)q(t) + 1}dt,$$

$$u_4(s) = p(s)\exp\int_0^s \frac{-\dot{p}(t)q(t)}{p(t)q(t) + 1}dt.$$

Taking into account $\dot{u}_2 = -\dot{u}_3p$, we get

$$u_2(s) = \exp\int_0^s \frac{-\dot{q}(t)p(t)}{p(t)q(t) + 1}dt,$$

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$$u_3(s) = q(s) \exp \int_0^s \frac{-\dot{q}(t)p(t)}{p(t)q(t)+1} dt.$$

For $A - B = 2$ we get

$$u_1 = (1+s)e^{-s}, \quad u_2 = (1-s)e^s, \quad u_3 = -Dse^s, \quad u_4 = -\frac{s}{D}e^{-s},$$

or in the original coordinates

$$x_1 = \cosh s - s \sinh s, \quad x_2 = -\sinh s + s \cosh s,$$

$$x_3 = -\frac{s}{2}\left(De^s + \frac{e^{-s}}{D}\right), \quad x_4 = -\frac{s}{2}\left(De^s - \frac{e^{-s}}{D}\right).$$

For $A - B = -2$ and for $|A - B| < 2$ one obtains the solution analogously in the timelike case $CD = 1$. Thus we get a two-parameter D and $A - B$ family of geodesics passing through the point $(1, 0, 0, 0)$. The parameters D and $A - B$ have a clear dynamical meaning. Namely,

$$D = -\dot{u}_3(0) = -(\dot{x}_3(0) + \dot{x}_4(0)), \quad C = \frac{1}{D} = -\dot{u}_4(0) = -(\dot{x}_3(0) - \dot{x}_4(0)),$$

and

$$A - B = \frac{\ddot{u}_3(0)}{\dot{u}_3(0)} = -\frac{\ddot{u}_4(0)}{\dot{u}_4(0)} = \frac{\ddot{x}_3(0) + \ddot{x}_4(0)}{\dot{x}_3(0) + \dot{x}_4(0)} = -\frac{\ddot{x}_3(0) - \ddot{x}_4(0)}{\dot{x}_3(0) - \dot{x}_4(0)}.$$

The spacelike case $CD = -1$ is treated in a similar way, but we omit awkward formulas.

Parametric solution. We present the parametric form of timelike and spacelike geodesics starting from the point $(1, 0, 0, 0)$. The forms of solutions with constant velocity coordinates (6.7) and (6.9) give us an idea of a suitable parametrization for geodesics with different causality.

  Timelike geodesics.

  We use the parametrization in a neighborhood of $(1, 0, 0, 0)$, given by

$$\begin{aligned}
x_1 &= \cos\phi \cosh\chi_1, \\
x_2 &= \sin\phi \cosh\chi_2, \\
x_3 &= \sin\phi \sinh\chi_2, \\
x_4 &= \cos\phi \sinh\chi_1,
\end{aligned} \tag{6.12}$$

where $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\chi_1, \chi_2 \in (\infty, \infty)$. We note that the timelike solution with constant velocity coordinates (6.7) followed from this parametrization if we set $\phi = -s$, $\chi_1 = 0$, and $\chi_2 = -\psi$. The vertical line Γ is obtained by setting $\phi = 0$, $\chi_1 = -s$, and $\chi_2 = 0$.

  In this parametrization the vector fields T, X, and Y admit the form

$$T = 2\cosh(\chi_1 - \chi_2)\partial_\phi + \partial_{\chi_1} \tan\phi \sinh(\chi_1 - \chi_2) + \partial_{\chi_2} \cotan\phi \sinh(\chi_1 - \chi_2),$$

$$X = 2\sinh(\chi_1 - \chi_2)\partial_\phi + \partial_{\chi_1} \tan\phi \cosh(\chi_1 - \chi_2) + \partial_{\chi_2} \cotan\phi \cosh(\chi_1 - \chi_2),$$

$$Y = \partial_{\chi_1} - \partial_{\chi_2}.$$

The vertical direction is given by the constant vector field Y. Let $c(s) = (\phi(s), \chi(s), \chi_2(s))$ be a curve starting at $c(0) = (0, 0, \chi_2(0))$. The horizontal coordinates (3.2) and (3.3) with respect to given parametrization are

$$\alpha = -\dot{\phi} \cosh(\chi_1 - \chi_2) + \frac{1}{2}(\dot{\chi}_1 + \dot{\chi}_2)\sin(2\phi)\sinh(\chi_1 - \chi_2),$$

$$\beta = \dot{\phi} \sinh(\chi_1 - \chi_2) + \frac{1}{2}(\dot{\chi}_1 + \dot{\chi}_2)\sin(2\phi)\cosh(\chi_1 - \chi_2).$$

Then, the square of the velocity vector $\dot{c}(s)$ is

$$-\alpha^2 + \beta^2 = -\dot{\phi}^2 + \frac{1}{4}(\dot{\chi}_1 + \dot{\chi}_2)^2 \sin^2(2\phi).$$

The speed is preserved along the geodesics and is equal to the initial value at the point $(1, 0, 0, 0)$, or in our parametrization $(0, 0, \chi_2(0))$. Therefore,

$$\langle \dot{c}(0), \dot{c}(0) \rangle = (-\alpha^2 + \beta^2)(0) = -\dot{\phi}^2(0),$$

and we obtain timelike geodesics starting from $(0, 0, \chi_2(0))$ if $\dot{\phi}(0) \neq 0$, and lightlike geodesics in the limiting case $\dot{\phi}(0) = 0$.

The Hamiltonian H associated with the operator

$$\mathcal{L} = \frac{1}{2}(-T^2 + X^2) = \frac{1}{2}(-4\partial_\phi^2 + \tan^2 \phi \partial_{\chi_1}^2 + \cotan^2 \phi \partial_{\chi_2}^2 + 2\partial_{\chi_1} \partial_{\chi_2}),$$

becomes

$$H(\phi, \chi_1, \chi_2, \psi, \xi_1, \xi_2) = \frac{1}{2}(-4\psi^2 + \xi_1^2 \tan^2 \phi + \xi_2^2 \cotan^2 \phi + 2\xi_1 \xi_2),$$

where we set $\partial_\phi = \psi$, $\partial_{\chi_1} = \xi_1$, and $\partial_{\chi_2} = \xi_2$.

The Hamiltonian system

$$\begin{aligned}
\dot{\chi}_1 &= \xi_1 \tan^2 \phi + \xi_2, \\
\dot{\chi}_2 &= \xi_2 \cotan^2 \phi + \xi_1, \\
\dot{\phi} &= -4\psi, \\
\dot{\xi}_1 &= 0, \\
\dot{\xi}_2 &= 0, \\
\dot{\psi} &= -\xi_1^2 \frac{\tan \phi}{\cos^2 \phi} + \xi_2^2 \frac{\cotan \phi}{\sin^2 \phi}.
\end{aligned} \qquad (6.13)$$

shows that ξ_1 and ξ_2 are constants. If both constants vanish, then we get

$$\dot{\chi}_1 = 0, \quad \dot{\chi}_2 = 0, \quad \dot{\phi} = -4\psi, \quad \dot{\psi} = 0,$$

which leads to the trivial solution (6.7). Since we are looking for a solution in a neighborhood of $(0, 0, \chi_2(0))$, we put $\xi_2 = 0$. Let us solve the Hamiltonian system (6.13) with the initial conditions

$$\phi(0) = 0, \quad \chi_1(0) = 0, \quad \chi_2(0) = \chi_2^{(0)}, \quad \psi(0) = \psi^{(0)}, \quad \xi_1(0) = \xi_1, \quad \xi_2(0) = 0.$$

From the third and from the last equations we get $\ddot{\phi} = -4\dot{\psi} = 4\xi_1^2 \frac{\tan \phi}{\cos^2 \phi}$. Multiplying by $\dot{\phi}$ and integrating we obtain

$$\dot{\phi}^2(s) = C^2 + 4\xi_1^2 \tan^2 \phi(s), \qquad C = \dot{\phi}^2(0) = 16\psi^2(0). \qquad (6.14)$$

Let us assume $C^2 > 0$. Simplifying (6.14), we come to

$$\frac{\cos \phi \, d\phi}{\sqrt{C^2 + (4\xi_1^2 - C^2) \sin^2 \phi}} = \pm ds.$$

According to the sign of $4\xi_1^2 - C^2$, one gets three different types of solutions.

Case 1: $4\xi_1^2 - C^2 = 0$. Integrating from 0 to some value of s we get the solution in the form $\sin \phi(s) = \pm |C| \, s$.

Case 2: $4\xi_1^2 - C^2 > 0$. The solution follows as

$$\sqrt{\frac{4\xi_1^2 - C^2}{C^2}} \sin \phi = \pm \sinh(s\sqrt{4\xi_1^2 - C^2}).$$

Case 3: $4\xi_1^2 - C^2 < 0$. The solution is obtained as

$$\sqrt{\frac{C^2 - 4\xi_1^2}{C^2}} \sin \phi = \pm \sin(s\sqrt{C^2 - 4\xi_1^2}).$$

In order to calculate the value of χ_1, we express $\tan^2 \phi$ from the Cases 1-3 and integrate the first equation of the Hamiltonian system. Observe that $\dot{\chi}_2 = \xi_1$ is constant and $\dot{\phi}(0) = -4\psi^{(0)} \neq 0$. The following theorem is proved.

Theorem 5 *The timelike geodesics starting from the point $\phi(0) = 0$, $\chi_1(0) = 0$, $\chi_2(0) = \chi_2^{(0)}$ with some $\dot\phi(0)$, a constant value of $\dot\chi_2$, and an arbitrary $\dot\chi_1(s)$ satisfy the following equations:*

if $4\dot\chi_2^2 = \dot\phi^2(0)$ then

- $\sin\phi(s) = \pm|C|s$,
- $\chi_1(s) = -\dot\chi_2 s + \dfrac{\dot\chi_2}{2\dot\phi(0)} \ln\left|\dfrac{1+\dot\phi(0)s}{1-\dot\phi(0)s}\right|$,
- $\chi_2(s) = \dot\chi_2 s + \chi_2^{(0)}$;

if $4\dot\chi_2^2 > \dot\phi^2(0)$ then

- $\sin\phi(s) = \pm\sqrt{\dfrac{\dot\phi^2(0)}{4\dot\chi_2^2-\dot\phi^2(0)}} \sinh\left(s\sqrt{4\dot\chi_2^2 - \dot\phi^2(0)}\right)$,
- $\chi_1(s) = -\dot\chi_2 s + \dot\chi_2 \int_0^s \dfrac{4\dot\chi_2^2-\dot\phi^2(0)}{4\dot\chi_2^2-\dot\phi^2(0)\cosh^2(s\sqrt{4\dot\chi_2^2-\dot\phi^2(0)})} ds$,
- $\chi_2(s) = \dot\chi_2 s + \chi_2^{(0)}$;

and if f $4\dot\chi_2^2 < \dot\phi^2(0)$ then

- $\sin\phi(s) = \pm\sqrt{\dfrac{\dot\phi^2(0)}{\dot\phi^2(0)-4\dot\chi_2^2}} \sin\left(s\sqrt{\dot\phi^2(0) - 4\dot\chi_2^2}\right)$,
- $\chi_1(s) = -\dot\chi_2 s + \dot\chi_2 \int_0^s \dfrac{\dot\phi^2(0)-4\dot\chi_2^2}{\dot\phi^2(0)\cos^2(s\sqrt{\dot\phi^2(0)-4\dot\chi_2^2})-4\dot\chi_2^2} ds$,
- $\chi_2(s) = \dot\chi_2 s + \chi_2^{(0)}$.

The integrals can be easily calculated and they involve trigonometric and hyperbolic functions, and depend on the relations between $4\dot\chi_2^2$, $\dot\phi^2(0)$.

Spacelike geodesics.

We use another parametrizaition in a neighborhood of $(1,0,0,0)$ suitable in this case

$$
\begin{aligned}
x_1 &= \cosh\phi\cosh\chi_1, \\
x_2 &= \sinh\phi\cosh\chi_2, \\
x_3 &= \sinh\phi\sinh\chi_2, \\
x_4 &= \cosh\phi\sinh\chi_1,
\end{aligned}
\tag{6.15}
$$

where $\phi, \chi_1, \chi_2 \in (-\infty, \infty)$. Observe that the spacelike solution with constant velocity coordinates (6.9) follows from this parametrization if we set $\phi = s$, $\chi_1 = 0$ and $\chi_2 = -\psi$. The vertical line Γ is obtained as previously, by setting $\phi = 0$, $\chi_1 = -s$, and $\chi_2 = 0$.

The vector fields T, X, and Y become

$$
\begin{aligned}
T &= 2\sinh(\chi_1 - \chi_2)\partial_\phi - \partial_{\chi_1}\tan\phi\cosh(\chi_1 - \chi_2) + \partial_{\chi_2}\cotan\phi\cosh(\chi_1 - \chi_2), \\
X &= 2\cosh(\chi_1 - \chi_2)\partial_\phi - \partial_{\chi_1}\tan\phi\sinh(\chi_1 - \chi_2) + \partial_{\chi_2}\cotan\phi\sinh(\chi_1 - \chi_2), \\
Y &= \partial_{\chi_1} - \partial_{\chi_2}.
\end{aligned}
$$

The vertical direction is again given by a constant vector field Y. Let $c(s) = (\phi(s), \chi(s), \chi_2(s))$ be a curve such that $c(0) = (0, 0, \chi_2(0))$. The horizontal coordinates (3.2) and (3.3) with respect to this parametrizaition are

$$
\begin{aligned}
\alpha &= \dot\phi\sinh(\chi_1 - \chi_2) - \frac{1}{2}(\dot\chi_1 + \dot\chi_2)\sinh(2\phi)\cosh(\chi_1 - \chi_2), \\
\beta &= \dot\phi\cosh(\chi_1 - \chi_2) - \frac{1}{2}(\dot\chi_1 + \dot\chi_2)\sinh(2\phi)\sinh(\chi_1 - \chi_2).
\end{aligned}
$$

Then the square of the velocity vector $\dot c$ is

$$
-\alpha^2 + \beta^2 = \dot\phi^2 - \frac{1}{4}(\dot\chi_1 + \dot\chi_2)^2\sinh^2(2\phi).
$$

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Since the speed is preserved along the geodesics, it is equal to $\dot{\phi}^2(0)$, and we obtain spacelike geodesics starting from $(0, 0, \chi_2(0))$ for $\dot{\phi}(0) \neq 0$.

The Hamiltonian H associated with the operator

$$\mathcal{L} = \frac{1}{2}(-T^2 + X^2) = \frac{1}{2}(4\partial_\phi^2 - \tanh^2 \phi \partial_{\chi_1}^2 - \cotanh^2 \phi \partial_{\chi_2}^2 + 2\partial_{\chi_1}\partial_{\chi_2})$$

becomes

$$H(\phi, \chi_1, \chi_2, \psi, \xi_1, \xi_2) = \frac{1}{2}(4\psi^2 - \xi_1^2 \tan^2 \phi - \xi_2^2 \cotan^2 \phi + 2\xi_1\xi_2),$$

where we set $\partial_\phi = \psi$, $\partial_{\chi_1} = \xi_1$, and $\partial_{\chi_2} = \xi_2$.

As in the previous case, the Hamiltonian system

$$\begin{aligned}
\dot{\chi}_1 &= -\xi_1 \tanh^2 \phi + \xi_2, \\
\dot{\chi}_2 &= -\xi_2 \cotanh^2 \phi + \xi_1, \\
\dot{\phi} &= 4\psi, \\
\dot{\xi}_1 &= 0, \\
\dot{\xi}_2 &= 0, \\
\dot{\psi} &= \xi_1^2 \frac{\tanh \phi}{\cosh^2 \phi} - \xi_2^2 \frac{\cotanh \phi}{\sinh^2 \phi}.
\end{aligned} \tag{6.16}$$

gives that ξ_1 and ξ_2 are constants. If both constants vanish, we get

$$\dot{\chi}_1 = 0, \quad \dot{\chi}_2 = 0, \quad \dot{\phi} = -4\psi, \quad \dot{\psi} = 0$$

which leads to the spacelike trivial solution. Setting $\xi_2 = 0$, we solve the Hamiltonian system (6.16) with the initial conditions

$$\phi(0) = 0, \quad \chi_1(0) = 0, \quad \chi_2(0) = \chi_2^{(0)}, \qquad \psi(0) = \psi^{(0)}, \quad \xi_1(0) = \xi_1, \quad \xi_2(0) = 0.$$

An analogue of (6.14) is

$$\dot{\phi}^2(s) = C^2 + 4\xi_1^2 \tanh^2 \phi(s), \qquad C = \dot{\phi}^2(0) = 16\psi^2(0) \neq 0. \tag{6.17}$$

Arguing as in the timelike case, we prove the following statement.

Theorem 6 *The spacelike geodesics starting from the point $\phi(0) = 0$, $\chi_1(0) = 0$, $\chi_2(0) = \chi_2^{(0)}$ with some $\dot{\phi}(0)$, a constant value of $\dot{\chi}_2$, and an arbitrary $\dot{\chi}_1(s)$ have the following equations:*

$$\sinh \phi(s) = \pm \sqrt{\frac{\dot{\phi}^2(0)}{\dot{\phi}^2(0) + 4\dot{\chi}_2^2}} \sinh(s\sqrt{\dot{\phi}^2(0) + 4\dot{\chi}_2^2}),$$

$$c\chi_1(s) = -\dot{\chi}_2 s + \frac{\dot{\chi}_2}{2|\dot{\chi}_2|} \operatorname{arccotanh}\left(\sqrt{\frac{\dot{\phi}^2(0) + 4\dot{\chi}_2^2}{4\dot{\chi}_2^2}} \cotan\left(s\sqrt{\dot{\phi}^2(0) + 4\dot{\chi}_2^2}\right)\right),$$

$$\chi_2(s) = \dot{\chi}_2 s + \chi_2^{(0)}.$$

7. Geodesics with respect to the distribution $\mathcal{D} = \operatorname{span}\{X, Y\}$

This case reveals the sub-Riemannian nature of such a distribution. In principle, one can easily modify the classical results from sub-Riemannian geometry (Chow-Rashevskii theorem, in particular). However we prefer to modify our own results proved in previous sections to show some particular features and to compare with the sub-Lorentzian case defined by the distribution $\mathcal{D} = \operatorname{span}\{T, X\}$.

Lemma 9 *A curve $c(s) = (x_1(s), x_2(s), x_3(s), x_4(s))$ is horizontal with respect to the distribution $\mathcal{D} = \operatorname{span}\{X, Y\}$, if and only if,*

$$x_2\dot{x}_1 - x_1\dot{x}_2 + x_4\dot{x}_3 - x_3\dot{x}_4 = 0 \qquad or \qquad \langle xJ, \dot{c}\rangle = 0. \tag{7.1}$$

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PROOF. The tangent vector to a curve $c(s) = (x_1(s), x_2(s), x_3(s), x_4(s))$ written in the left-invariant basis is of the form

$$\dot{c}(s) = \alpha T + \beta X + \gamma Y.$$

Then

$$\alpha = \langle \dot{c}, T \rangle = \mathcal{I}\dot{c} \cdot T = \dot{x}_1 x_2 - \dot{x}_2 x_1 + \dot{x}_3 x_4 - \dot{x}_4 x_3.$$

We conclude that $\alpha = 0$, if and only if, (7.1) holds. $\square$

In this case a curve is horizontal, if and only if, its velocity vector is orthogonal to the vector field T. The left-invariant coordinates $\beta(s)$ and $\gamma(s)$ of a horizontal curve $c(s) = (x_1(s), x_2(s), x_3(s), x_4(s))$ are

$$\beta = \langle \dot{c}, X \rangle = -x_3 \dot{x}_1 - x_4 \dot{x}_2 + x_1 \dot{x}_3 + x_2 \dot{x}_4 = \langle xE_1, \dot{c} \rangle. \tag{7.2}$$

$$\gamma = \langle \dot{c}, Y \rangle = -x_4 \dot{x}_1 + x_3 \dot{x}_2 - x_2 \dot{x}_3 + x_1 \dot{x}_4 = \langle xE_2, \dot{c} \rangle. \tag{7.3}$$

The form $w = -x_2 dx_1 + x_1 dx_2 - x_4 dx_3 + x_3 dx_4 = -\langle xJ, dx \rangle$ is a contact form for the horizontal distribution $\mathcal{D} = \text{span}\{X, Y\}$. Indeed

$$w(N) = 0, \quad w(T) = 1, \quad w(X) = 0, \quad w(Y) = 0.$$

Thus, $\ker w = \text{span}\{N, X, Y\}$, The horizontal distribution can be defined as follows

$$\mathcal{D} = \{V \in TAdS : w(V) = 0\}, \quad \text{or} \quad \mathcal{D} = \ker w \cap TAdS.$$

The length $l(c)$ of a horizontal curve $c(s) : [0, 1] \to AdS$ is given by

$$l(c) = \int_0^1 \langle \dot{c}(s), \dot{c}(s) \rangle^{1/2} \, ds = \int_0^1 \left(\beta^2(s) + \gamma^2(s) \right)^{1/2} ds.$$

The restriction of the non-degenerate metric $\langle \cdot, \cdot \rangle$ onto the horizontal distribution $\mathcal{D} \subset TAdS$ gives a positive-definite metric that we still denote by $\langle \cdot, \cdot \rangle_{\mathcal{D}}$. Thus from now on, we shall work only with one type of the curves (that we shall call simply horizontal curves), since the horizontality condition requires the vanishing coordinate function of the vector field T.

7.1. *Existence of horizontal curves*

The following theorem is an analogue to Theorem 1 proved for the distribution $\mathcal{D} = \text{span}\{T, X\}$ in Section 4.

Theorem 7 *Let P, $Q \in AdS$ be arbitrary given points. Then there is a smooth horizontal curve connecting P with Q.*

PROOF. We use parametrisation (4.1), in which the horizontality condition for a curve $c(s)$ is expressed by (4.3) as

$$\dot{\psi} + \dot{\varphi} \cosh 2\theta = 0.$$

This equation is to be sold for the initial conditions

$$c(0) = P, \quad \text{or} \quad \varphi(0) = \varphi_0, \quad \psi(0) = \psi_0, \quad \theta(0) = \theta_0,$$

$$c(1) = Q, \quad \text{or} \quad \varphi(1) = \varphi_1, \quad \psi(1) = \psi_1, \quad \theta(1) = \theta_1.$$

Let $\psi = \psi(s)$ be a smooth arbitrary function with $\dot{\psi}(0) = \lim_{s \to 0^+} \dot{\psi}(s)$ and $\dot{\psi}(1) = \lim_{s \to 1^-} \dot{\psi}(s)$. Set $2\theta(s) = \text{arccosh}\, p(s)$. Then the equation (4.3) admits the form

$$\dot{\varphi} = -\frac{\dot{\psi}}{\cosh 2\theta} = -\frac{\dot{\psi}}{p(s)} \quad \Rightarrow \quad \varphi(s) = -\int_0^s \frac{\dot{\psi}(s)\, ds}{p(s)} + \varphi(0).$$

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Denote $q(s) = \frac{\dot{\psi}(s)}{p(s)}$. Since $q(0) = \frac{\dot{\psi}(0)}{\cosh 2\theta_0}$, $q(1) = \frac{\dot{\psi}(1)}{\cosh 2\theta_1}$, and $\int_0^1 q(s)\,ds = \varphi_0 - \varphi_1$ applying Lemma 5 we conclude that there exists such a smooth function $q(s)$. The function $p(s)$ is found as $p(s) = \frac{\dot{\psi}(s)}{q(s)}$. We get a curve $c(s) = (\varphi(s), \psi(s), \theta(s))$ with

$$
\begin{aligned}
\psi &= \psi(s), \\
\varphi(s) &= -\int_0^s \frac{\dot{\psi}(s)\,ds}{p(s)} + \varphi(0), \\
\theta(s) &= \frac{1}{2}\operatorname{arccosh} p(s).
\end{aligned}
$$

$\square$

Remark 3 *Observe that in the general Chow-Rashevskii theorem smoothness was not concluded.*

Theorem 8 *Given two arbitrary points $P = P(\varphi_0, \psi_0, \theta_0)$ and $Q = Q(\varphi_1, \psi_1, \theta_0)$ with $2\theta_0 = \operatorname{arccosh}\frac{\psi_1 - \psi_0}{\varphi_0 - \varphi_1}$, there is a horizontal curve with the constant θ-coordinate connecting P with Q.*

PROOF. If the θ-coordinate is constant, then the governing equation is

$$
\dot{\psi} = -\dot{\varphi}\cosh 2\theta_0 \quad \Rightarrow \quad \psi(s) = -\varphi(s)\cosh 2\theta_0 + C.
$$

Applying the initial conditions

$$
c(0) = (\varphi_0, \psi_0, \theta_0), \quad \text{and} \quad c(1) = (\varphi_1, \psi_1, \theta_0),
$$

we find

$$
2\theta_0 = \operatorname{arccosh}\left(\frac{\psi_1 - \psi_0}{\varphi_0 - \varphi_1}\right), \qquad C = \psi_0 + \varphi_0 \frac{\psi_1 - \psi_0}{\varphi_0 - \varphi_1}.
$$

Therefore, for any parameter φ, the horizontal curve

$$
c(s) = \left(\varphi, \psi_0 + (\varphi(0) - \varphi)\frac{\psi_1 - \psi_0}{\varphi_0 - \varphi_1}, \theta_0\right), \qquad 2\theta_0 = \operatorname{arccosh}\frac{\psi_1 - \psi_0}{\varphi_0 - \varphi_1},
$$

joins the points $P = P(\varphi_0, \psi_0, \theta_0)$ and $Q = Q(\varphi_1, \psi_1, \theta_0)$. $\square$

7.2. *Lagrangian formalism*

Dealing with $\mathcal{D} = \operatorname{span}\{X, Y\}$ and a positive-definite metric $\langle \cdot, \cdot \rangle_\mathcal{D}$ on it, one might compare with the geometry generated by the sub-Riemannian distribution on sphere S^3 in [4]. The minimizing length curve can be found by minimizing the action integral

$$
S = \frac{1}{2}\int_0^1 (\beta^2(s) + \gamma^2(s))\,ds
$$

under the non-holonomic constrain $\alpha = \langle \dot{c}, xJ \rangle = 0$. The corresponding Lagrangian is

$$
L(c, \dot{c}) = \frac{1}{2}(\beta^2(s) + \gamma^2(s)) + \lambda(s)\alpha(s). \tag{7.4}
$$

The extremal curve is given by the solution of the Euler-Lagrange system (5.1) with the Lagrangian (7.4).

Let us make some preparatory calculations. Write the system (5.1) for the Lagrangian (7.4) as the follows

$$
\begin{aligned}
2\beta\dot{x}_3 + 2\gamma\dot{x}_4 - 2\lambda\dot{x}_2 + \dot{\beta}x_3 + \dot{\gamma}x_4 - \dot{\lambda}x_2 &= 0, \\
2\beta\dot{x}_4 - 2\gamma\dot{x}_3 + 2\lambda\dot{x}_1 + \dot{\beta}x_4 - \dot{\gamma}x_3 + \dot{\lambda}x_1 &= 0, \\
-2\beta\dot{x}_1 + 2\gamma\dot{x}_2 - 2\lambda\dot{x}_4 - \dot{\beta}x_1 + \dot{\gamma}x_2 - \dot{\lambda}x_4 &= 0, \\
-2\beta\dot{x}_2 - 2\gamma\dot{x}_1 + 2\lambda\dot{x}_3 - \dot{\beta}x_2 - \dot{\gamma}x_1 + \dot{\lambda}x_3 &= 0.
\end{aligned}
$$

Multiply the equations by x_3, x_4, x_1, and x_2, respectively and sum them up. We get

$$2\beta\langle \dot{c}, N\rangle - 2\gamma\langle \dot{c}, T\rangle - 2\lambda\langle \dot{c}, Y\rangle - \dot{\beta} + 0\dot{\gamma} + 0\dot{\lambda} = 0 \quad \Rightarrow \quad \dot{\beta} = 2\lambda\gamma,$$
$$2\beta\langle \dot{c}, T\rangle - 2\gamma\langle \dot{c}, N\rangle + 2\lambda\langle \dot{c}, X\rangle + 0\dot{\beta} - \dot{\gamma} + 0\dot{\lambda} = 0 \quad \Rightarrow \quad \dot{\gamma} = 2\lambda\beta.$$

Let us consider two cases.

Case $\lambda(s) = 0$. In this case equation (7.2) admits the form

$$\dot{\beta} = 0, \qquad \dot{\gamma} = 0, \tag{7.5}$$

and we deduce the following theorem.

Theorem 9 *There are horizontal geodesics with the following properties:*
1. The coordinates $\alpha = \langle \dot{c}, T\rangle = 0$, $\beta = \langle \dot{c}, X\rangle$, and $\gamma = \langle \dot{c}, Y\rangle$ are constant;
2. The length $|\dot{c}|$ along the geodesics;
3. The angles between the velocity vector and horizontal frame is constant along along the geodesic.

PROOF. Taking into account the solution of (7.5), we denote $\beta(s) = \beta$ and $\gamma(s) = \gamma$. Then the length of the velocity vector $|\dot{c}| = \sqrt{\beta^2 + \gamma^2}$ is constant.

Since $\langle \dot{c}, X\rangle = \langle \dot{c}, X\rangle_{\mathcal{D}} = |\dot{c}|_{\mathcal{D}}|X|_{\mathcal{D}}\cos(\angle\dot{c}, X)$, $\langle \dot{c}, Y\rangle = \langle \dot{c}, Y\rangle_{\mathcal{D}} = |\dot{c}|_{\mathcal{D}}|Y|_{\mathcal{D}}\cos(\angle\dot{c}, Y)$, we have

$$\cos(\angle\dot{c}, X) = \frac{\beta}{\sqrt{\beta^2 + \gamma^2}}, \quad \cos(\angle\dot{c}, Y) = \frac{\gamma}{\sqrt{\beta^2 + \gamma^2}},$$

that proves the third assertion. $\square$

Case $\lambda(s) \neq 0$.
Theorem 10 *There are horizontal geodesics with the following properties:*
1. The velocity vector $|\dot{c}|$ of a geodesic is constant along the geodesic;
2. The angles between the velocity vector and the horizontal frame are given by

$$\angle\dot{c}, X = cs + \theta_0, \qquad \angle\dot{c}, Y = \frac{\pi}{2} - cs + \theta_0.$$

PROOF. Since

$$\dot{\beta} = 2\lambda\gamma, \quad \dot{\gamma} = 2\lambda\beta \tag{7.6}$$

implies $\frac{d}{ds}(\beta^2 + \gamma^2) = 0$, we conclude, that the length of the velocity vector $|\dot{c}|$ is constant. Taking into account positivity of $\beta^2 + \gamma^2$ let us denote it by r^2. Set $\beta = r\cos\theta(s)$ and $\gamma = r\sin\theta(s)$. Substituting them in (7.6), we get

$$\dot{\theta}(s) = 2\lambda(s) \quad \Rightarrow \quad \theta(s) = 2\int \lambda(s)\,ds + \theta_0.$$

Let us find the function $\lambda(s)$. Observe that

$$\beta^2 + \gamma^2 = -\dot{x}_1^2 - \dot{x}_2^2 + \dot{x}_3^2 + \dot{x}_4^2.$$

It can be shown similarly to the proof of Proposition 2, having $\alpha = \delta = 0$. By the direct calculation (see also Proposition 2) we show that

$$\langle \ddot{c}, T\rangle = \frac{d}{ds}\langle \dot{c}, T\rangle = 0.$$

Now, we consider an equivalent to (7.4) extremal problem with the Lagrangian

$$\widehat{L}(c, \dot{c}) = \frac{1}{2}\left(-\dot{x}_1^2 - \dot{x}_2^2 + \dot{x}_3^2 + \dot{x}_4^2\right) + \lambda(s)\langle \dot{c}, T\rangle. \tag{7.7}$$

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The Euler-Lagrange system admits the form

$$-\ddot{x}_1 = -2\lambda\dot{x}_2 - \dot{\lambda}x_2,$$
$$-\ddot{x}_2 = 2\lambda\dot{x}_1 + \dot{\lambda}x_1,$$
$$\ddot{x}_3 = -2\lambda\dot{x}_4 - \dot{\lambda}x_4,$$
$$\ddot{x}_4 = 2\lambda\dot{x}_3 + \dot{\lambda}x_3.$$

Multiplying these equations by $x_2, -x_1, -x_4, x_3$ respectively and then, summing them up, we obtain

$$-\langle \ddot{c}, T \rangle = 2\lambda\langle \dot{c}, N \rangle - \dot{\lambda}.$$

This allows us to conclude, that the function $\lambda(s)$ is constant along the solution of the Euler-Lagrange equation that yields the second assertion of the theorem. $\square$

7.3. *Hamiltonian formalism*

The sub-Laplacian is $\mathcal{L} = X^2 + Y^2$ and the corresponding Hamiltonian function is

$$H(x,\xi) = \frac{1}{2}\Big(\big(x_3\xi_1 + x_4\xi_2 + x_1\xi_3 + x_2\xi_4\big)^2 + \big(x_4\xi_1 - x_3\xi_2 - x_2\xi_3 + x_1\xi_4\big)^2 \Big) = \frac{1}{2}(\varsigma^2 + \kappa^2).$$

The Hamiltonian system is written as

$$\begin{aligned}
\dot{x} &= \frac{\partial H}{\partial \xi} &&= \varsigma x E_1 + \kappa x E_2, \\
\dot{\xi} &= -\frac{\partial H}{\partial x} &&= -\varsigma \xi E_1 - \kappa \xi E_2,
\end{aligned} \tag{7.8}$$

As in the previous section we are able to prove the following proposition.

Proposition 3 *The solution of the Hamiltonian system is a horizontal curve and*

$$\varsigma = \beta, \qquad \kappa = \gamma.$$

Corollary 2 *The Hamiltonian function is the energy* $H(x,\xi) = \frac{1}{2}(\beta^2 + \gamma^2)$.

7.4. *Geodesics with constant horizontal coordinates.*

In this section we consider constant horizontal coordinates β and γ. Making use of Proposition 3 we write the first line of the Hamiltonian system (7.8) in the form.

$$\begin{aligned}
\dot{x}_1 &= \beta x_3 + \gamma x_4, \\
\dot{x}_2 &= \beta x_4 - \gamma x_3, \\
\dot{x}_3 &= \beta x_1 - \gamma x_2, \\
\dot{x}_4 &= \beta x_2 + \gamma x_1,
\end{aligned} \tag{7.9}$$

We give an explicit picture for the base point $(1,0,0,0)$. Without lost of generality, let us assume that $\beta^2 + \gamma^2 = 1$, $\beta = \cos\psi$, $\gamma = \sin\psi$, where ψ is a constant.

The Hamiltonian system (7.9) written for constant β and γ is reduced to a second-order differential equation

$$\ddot{x}_k = x_k, \quad k = 1,\ldots 4. \tag{7.10}$$

The general solution is given in the hyperbolic basis as $x_k = A_k \cosh s + B_k \sinh s$. The initial condition $x(0) = (1,0,0,0)$ defines the coefficients A_k by $A_1 = 1$, $A_2 = A_3 = A_k = 0$. Returning back to the

first-order system (7.9) we calculate the coefficients B_k as $B_1 = 0$, $B_2 = 0$, $B_3 = \beta$, $B_4 = \gamma$. Finally, the solution is

$$x_1 = \cosh s, \quad x_2 \equiv 0, \quad x_3 = \cos\psi \sinh s, \quad x_4 = \sin\psi \sinh s. \tag{7.11}$$

Varying ψ they sweep out the two-sheet hyperboloid $x_1^2 - x_3^2 - x_4^2 = 1$ in $\mathbb{R}^3$. We use only one sheet containing the point $(1, 0, 0, 0)$. Geodesics are hyperbolas passing this point.

The vertical line corresponds to the vanishing horizontal velocity (β, γ) and with the constant value $\alpha = 1$, passing the base point $(1, 0, 0, 0)$. The solution is

$$\Gamma(s) = (\cos s, \sin s, 0, 0).$$

The vertical line (circle) Γ meets the surface (7.11) at the point $(1,0,0,0)$ orthogonally with respect to the scalar product in $\mathbb{R}^{2,2}$.

7.5. *Geodesics with non-constant horizontal coordinates.*

If the horizontal coordinates are not constant, then we must solve the Hamiltonian system generated by the above Hamiltonian.

Solution in the Cartesian coordinates. Fix the initial point $x^{(0)} = (1, 0, 0, 0)$. In the Cartesian case it is convenient to introduce complex coordinates $z = x_1 + ix_2$, $w = x_3 + ix_4$, $\varphi = \xi_1 + i\xi_2$, and $\psi = \xi_3 + i\xi_4$. Hence, the Hamiltonian admits the form $H = |z\bar{\psi} + \bar{w}\varphi|^2$. The corresponding Hamiltonian system becomes

$$\begin{aligned}
\dot{z} &= w(z\bar{\psi} + \bar{w}\varphi), & z(0) &= 1, \\
\dot{w} &= z(\bar{z}\psi + w\bar{\varphi}), & w(0) &= 0, \\
\dot{\bar{\varphi}} &= -\bar{\psi}(\bar{z}\psi + w\bar{\varphi}), & \bar{\varphi}(0) &= A - iB, \\
\dot{\bar{\psi}} &= -\bar{\varphi}(z\bar{\psi} + \bar{w}\varphi), & \bar{\psi}(0) &= C - iD.
\end{aligned}$$

Here the constants $A, B, C,$ and D have the following dynamical meaning: $\dot{w}(0) = C + iD$, and $2B = i\ddot{w}(0)/\dot{w}(0)$. This complex Hamiltonian system has the first integrals

$$\begin{aligned}
z\psi + w\varphi &= C + iD, \\
z\bar{\varphi} + w\bar{\psi} &= A - iB,
\end{aligned}$$

and we have $|z|^2 - |w|^2 = 1$ and $H = C^2 + D^2 = 1$ as an additional normalization. Therefore,

$$\begin{aligned}
\varphi &= z(A + iB) - \bar{w}(C + iD), \\
\psi &= \bar{z}(C + iD) - w(A + iB).
\end{aligned}$$

Let us introduce an auxiliary function $p = \bar{w}/z$. Then substituting φ and ψ in the Hamiltonian system we get

$$p(s) = -(C - iD)\frac{1 + e^{-2s\sqrt{1 - B^2}}}{\sqrt{1 - B^2} - iB + (\sqrt{1 - B^2} - iB)e^{-2s\sqrt{1 - B^2}}}.$$

Taking into account that $\dot{z}\bar{z} = w\dot{\bar{w}}$, we get the solution for $B \neq 1$

$$z(s) = \exp\int_0^s \frac{\bar{p}(t)\dot{p}(t)}{1 - |p(t)|^2}\,dt,$$

and

$$w(s) = \bar{p}(s)\exp\int_0^s \frac{p(t)\dot{\bar{p}}(t)}{1 - |p(t)|^2}\,dt.$$

For $B = 1$ the solution is

$$z(s) = (1 + is)e^{is}, \quad w(s) = s(C + iD)e^{-is}.$$

Parametric solution. Let us present the parametric solution in this case. We use the parametrization in a neighbourhood of $(1, 0, 0, 0)$ given by

$$
\begin{aligned}
x_1 &= \cos \chi_1 \cosh \phi, \\
x_2 &= \sin \chi_1 \cosh \phi, \\
x_3 &= \cos \chi_2 \sinh \phi, \\
x_4 &= \sin \chi_2 \sinh \phi,
\end{aligned}
\tag{7.12}
$$

where $\phi \in (-\infty, \infty)$, $\chi_1, \chi_2 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. We observe that the solution with constant velocity coordinates (7.11) follows from this parameterization when we set $\phi = s$, $\chi_1 = 0$, and $\chi_2 = \psi$. The vertical line (circle) Γ is obtained by setting $\phi = 0$, $\chi_1 = s$, and $\chi_2 = 0$.

In this parametrization, the vector fields T, X, and Y admit the form

$$
\begin{aligned}
T &= \partial_{\chi_1} - \partial_{\chi_2}, \\
X &= 2\cos(\chi_1 - \chi_2)\partial_\phi - \partial_{\chi_1} \tanh \phi \sin(\chi_1 - \chi_2) + \partial_{\chi_2} \operatorname{cotanh} \phi \sin(\chi_1 - \chi_2), \\
Y &= 2\sin(\chi_1 - \chi_2)\partial_\phi - \partial_{\chi_1} \tanh \phi \cos(\chi_1 - \chi_2) + \partial_{\chi_2} \operatorname{cotanh} \phi \cos(\chi_1 - \chi_2).
\end{aligned}
$$

The vertical direction is given by the constant vector field T.

The Hamiltonian H associated with the operator

$$
\mathcal{L} = \frac{1}{2}(X^2 + Y^2) = \frac{1}{2}(4\partial_\phi^2 + \tanh^2 \phi \, \partial_{\chi_1}^2 + \operatorname{cotanh}^2 \phi \, \partial_{\chi_2}^2 - 2\partial_{\chi_1}\partial_{\chi_2})
$$

is given as

$$
H(\phi, \chi_1, \chi_2, \psi, \xi_1, \xi_2) = \frac{1}{2}(4\psi^2 + \xi_1^2 \tanh^2 \phi + \xi_2^2 \operatorname{cotanh}^2 \phi - 2\xi_1\xi_2),
$$

where we set $\partial_\phi = \psi$, $\partial_{\chi_1} = \xi_1$, and $\partial_{\chi_2} = \xi_2$.

Description of geodesics is collected in the following theorem.

Theorem 11 *The geodesics starting from the point $\phi(0) = 0$, $\chi_1(0) = 0$, $\chi_2(0) = \chi_2^{(0)}$ with some $\dot\phi(0)$, a constant value of $\dot\chi_2$, and an arbitrary $\dot\chi_1(s)$ have the following equations.*

If $4\dot\chi_2^2 = \dot\phi^2(0)$ then
- $\sinh \phi(s) = \pm|C|s$,
- $\chi_1(s) = \dot\chi_2 s - \frac{\dot\chi_2}{\dot\phi(0)} \arctan \dot\phi(0)s$,
- $\chi_2(s) = -\dot\chi_2 s + \chi_2^{(0)}$.

If $4\dot\chi_2^2 > \dot\phi^2(0)$ then
- $\sinh \phi(s) = \pm\sqrt{\dfrac{\dot\phi^2(0)}{4\dot\chi_2^2 - \dot\phi^2(0)}} \sin\left(s\sqrt{4\dot\chi_2^2 - \dot\phi^2(0)}\right)$,
- $\chi_1(s) = \dot\chi_2 s - \dfrac{\dot\chi_2}{2|\dot\chi_2|} \arctan\left(\sqrt{\dfrac{4\dot\chi_2^2 - \dot\phi^2(0)}{4\dot\chi_2^2}} \tan\left(s\sqrt{4\dot\chi_2^2 - \dot\phi^2(0)}\right)\right)$,
- $\chi_2(s) = -\dot\chi_2 s + \chi_2^{(0)}$.

If $4\dot\chi_2^2 < \dot\phi^2(0)$ then
- $\sinh \phi(s) = \pm\sqrt{\dfrac{\dot\phi^2(0)}{\dot\phi^2(0) - 4\dot\chi_2^2}} \sinh\left(s\sqrt{\dot\phi^2(0) - 4\dot\chi_2^2}\right)$,
- $\chi_1(s) = \dot\chi_2 s - \dfrac{\dot\chi_2}{2|\dot\chi_2|} \arctan\left(\sqrt{\dfrac{\dot\phi^2(0) - 4\dot\chi_2^2}{4\dot\chi_2^2}} \cotan\left(s\sqrt{4\dot\chi_2^2 - \dot\phi^2(0)}\right)\right)$,
- $\chi_2(s) = -\dot\chi_2 s + \chi_2^{(0)}$.

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